Stability
Recurrent Networks

Nonlinear Recurrent Network

\[ da(t)/dt = g(a(t), p(t), t) \]
Types of Stability

A ball bearing, with dissipative friction, in a gravity field:

- Asymptotically Stable
- Stable in the Sense of Lyapunov
- Unstable
In the Hopfield network we want the prototype patterns to be stable points with large basins of attraction.
Lyapunov Stability

\[ \frac{d}{dt} \mathbf{a}(t) = \mathbf{g}(\mathbf{a}(t), \mathbf{p}(t), t) \]

**Equilibrium Point:**
An equilibrium point is a point \( \mathbf{a}^* \) where \( \frac{d\mathbf{a}}{dt} = 0 \).

**Stability (in the sense of Lyapunov):**
The origin is a stable equilibrium point if for any given value \( \varepsilon > 0 \) there exists a number \( \delta(\varepsilon) > 0 \) such that if \( \|\mathbf{a}(0)\| < \delta \), then the resulting motion, \( \mathbf{a}(t) \), satisfies \( \|\mathbf{a}(t)\| < \varepsilon \) for \( t > 0 \).
Asymptotic Stability:

The origin is an asymptotically stable equilibrium point if there exists a number $\delta > 0$ such that if $\|a(0)\| < \delta$, then the resulting motion, $a(t)$, satisfies $\|a(t)\| \to 0$ as $t \to \infty$. 

$$\frac{d}{dt} a(t) = g(a(t), p(t), t)$$
Definite Functions

Positive Definite:
A scalar function $V(a)$ is positive definite if $V(0) = 0$ and $V(a) > 0$ for $a \neq 0$.

Positive Semidefinite:
A scalar function $V(a)$ is positive semidefinite if $V(0) = 0$ and $V(a) \geq 0$ for all $a$. 
Lyapunov Stability Theorem

\[ \frac{d\mathbf{a}}{dt} = g(\mathbf{a}) \]

Theorem 1: Lyapunov Stability Theorem

If a positive definite function \( V(\mathbf{a}) \) can be found such that \( dV(\mathbf{a})/dt \) is negative semidefinite, then the origin \( (\mathbf{a} = 0) \) is stable for the above system. If a positive definite function \( V(\mathbf{a}) \) can be found such that \( dV(\mathbf{a})/dt \) is negative definite, then the origin \( (\mathbf{a} = 0) \) is asymptotically stable. In each case, \( V(\mathbf{a}) \) is called a Lyapunov function of the system.
Pendulum Example

State Variable Model

\[ a_1 = \theta \]
\[ \frac{da_1}{dt} = a_2 \]
\[ a_2 = \frac{d\theta}{dt} \]
\[ \frac{da_2}{dt} = -\frac{g}{l} \sin(a_1) - \frac{c}{ml} a_2 \]

\[ ml \frac{d^2 \theta}{dt^2} + c \frac{d\theta}{dt} + mg \sin(\theta) = 0 \]
Equilibrium Point

Check: \( a = 0 \)

\[
\frac{da_1}{dt} = a_2 = 0
\]

\[
\frac{da_2}{dt} = -\frac{g}{l} \sin(a_1) - \frac{c}{ml}a_2 = -\frac{g}{l} \sin(0) - \frac{c}{ml}(0) = 0
\]

Therefore the origin is an equilibrium point.
Lyapunov Function (Energy)

\[ V(\mathbf{a}) = \frac{1}{2}ml^2(a_2)^2 + mgl(1 - \cos(a_1)) \]  

\begin{align*}
\text{Kinetic Energy} & \quad \text{Potential Energy} \\
\frac{d}{dt}V(\mathbf{a}) & = \nabla V(\mathbf{a})^T g(\mathbf{a}) = \frac{\partial V}{\partial a_1} \left( \frac{da_1}{dt} \right) + \frac{\partial V}{\partial a_2} \left( \frac{da_2}{dt} \right) \\
\frac{d}{dt}V(\mathbf{a}) & = (mgl \sin(a_1))a_2 + (ml^2a_2) \left( -\frac{g}{l} \sin(a_1) - \frac{c}{ml} a_2 \right) \\
\frac{d}{dt}V(\mathbf{a}) & = -cl(a_2)^2 \leq 0
\end{align*}

The derivative is negative semidefinite, which proves that the origin is stable in the sense of Lyapunov (at least).
Numerical Example

\[ g = 9.8, \quad m = 1, \quad l = 9.8, \quad c = 1.96 \]

\[ \frac{d a_1}{d t} = a_2 \quad \quad \frac{d a_2}{d t} = -\sin(a_1) - 0.2a_2 \]

\[ V = (9.8)^2 \left[ \frac{1}{2} (a_2)^2 + (1 - \cos(a_1)) \right] \quad \quad \frac{dV}{dt} = -(19.208)(a_2)^2 \]
\[ a(0) = \begin{bmatrix} 1.3 \\ 1.3 \end{bmatrix} \]
Lyapunov Function

Let $V(a)$ be a continuously differentiable function from $\mathbb{R}^n$ to $\mathbb{R}$. If $G$ is any subset of $\mathbb{R}^n$, we say that $V$ is a Lyapunov function on $G$ for the system $da/dt = g(a)$ if

$$\frac{dV(a)}{dt} = (\nabla V(a))^T g(a)$$

does not change sign on $G$.

Set $Z$

$$Z = \{a: dV(a)/dt = 0, \ a \text{ in the closure of } G\}$$
Invariant Set

A set of points in $\mathbb{R}^n$ is invariant with respect to $\frac{da}{dt} = g(a)$ if every solution of $\frac{da}{dt} = g(a)$ starting in that set remains in the set for all time.

Set $L$

$L$ is defined as the largest invariant set in $Z$. 
Theorem 2: Lasalle’s Invariance Theorem

If $V$ is a Lyapunov function on $G$ for $\frac{d a}{dt} = g(a)$, then each solution $a(t)$ that remains in $G$ for all $t > 0$ approaches $L^\circ = L \cup \{\infty\}$ as $t \to \infty$. ($G$ is a basin of attraction for $L$, which has all of the stable points.) If all trajectories are bounded, then $a(t) \to L$ as $t \to \infty$.

Corollary 1: Lasalle’s Corollary

Let $G$ be a component (one connected subset) of

$$\Omega_\eta = \{a: V(a) < \eta\}.$$

Assume that $G$ is bounded, $dV(a)/dt \leq 0$ on the set $G$, and let the set $L^\circ = \text{closure}(L \cup G)$ be a subset of $G$. Then $L^\circ$ is an attractor, and $G$ is in its region of attraction.
\[ \Omega_{100} = \{ \mathbf{a} : V(\mathbf{a}) \leq 100 \} \]

\[ G = \text{One component of } \Omega_{100}. \]
$$Z = \{ \mathbf{a}: \frac{dV(\mathbf{a})}{dt} = 0, \ \mathbf{a} \text{ in the closure of } G \} = \{ \mathbf{a}: a_2 = 0, \ \mathbf{a} \text{ in the closure of } G \}$$

$$L = \{ \mathbf{a}: \mathbf{a} = 0 \}$$
For this choice of $G$ we can say little about where the trajectory will converge.
Pendulum Trajectory

![Pendulum Trajectory Diagram]
We want $G$ to be as large as possible, because that will indicate the region of attraction. However, we want to choose $V$ so that the set $Z$, which will contain the attractor set, is as small as possible.

$V = 0$ is a Lyapunov function for all of $\mathbb{R}^n$, but it gives no information since $Z = \mathbb{R}^n$.

If $V_1$ and $V_2$ are Lyapunov functions on $G$, and $dV_1/dt$ and $dV_2/dt$ have the same sign, then $V_1 + V_2$ is also a Lyapunov function, and $Z = Z_1 \cap Z_2$. If $Z$ is smaller than $Z_1$ or $Z_2$, then $V$ is a “better” Lyapunov function than either $V_1$ or $V_2$. $V$ is always at least as good as either $V_1$ or $V_2$. 