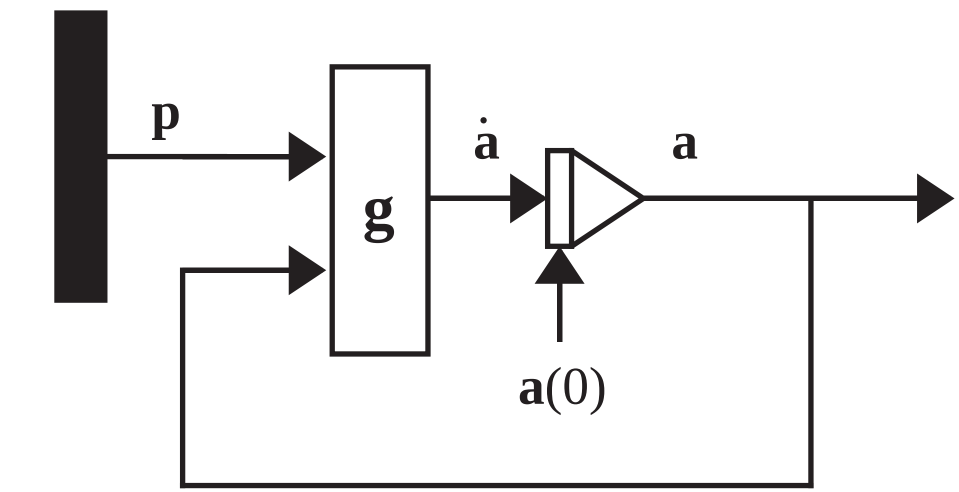
Stability

Recurrent Networks

Nonlinear Recurrent Network



$$d\mathbf{a}(t)/dt = \mathbf{g}(\mathbf{a}(t), \mathbf{p}(t), t)$$

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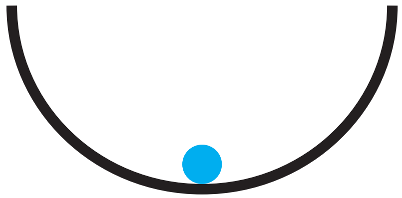
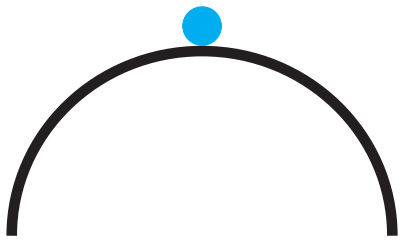
Types of Stability

A ball bearing, with dissipative friction, in a gravity field:



Asymptotically Stable

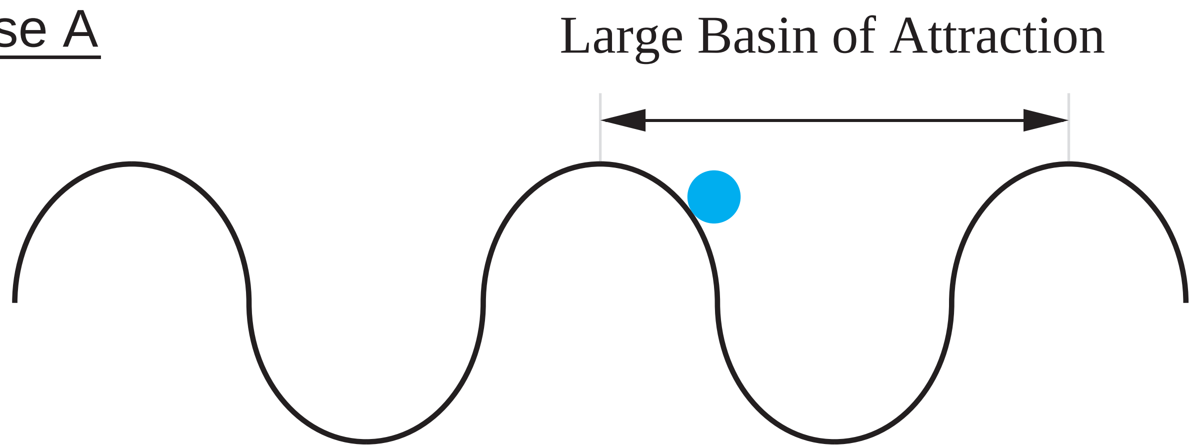
Stable in the Sense of Lyapunov



Unstable

Basins of Attraction

Case A



Case B

In the Hopfield network we want the prototype patterns to be stable points with large basins of attraction.

Lyapunov Stability

$$\frac{d}{dt}\mathbf{a}(t) = \mathbf{g}(\mathbf{a}(t), \mathbf{p}(t), t)$$

Eqilibrium Point:

An equilibrium point is a point $\mathbf{a}^*$ where $d\mathbf{a}/dt = \mathbf{0}$.

Stability (in the sense of Lyapunov):

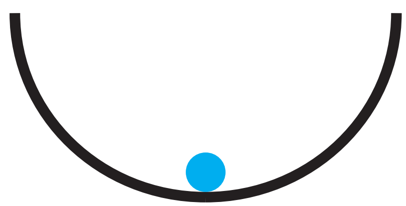
The origin is a stable equilibrium point if for any given value $\varepsilon > 0$ there exists a number $\delta(\varepsilon) > 0$ such that if $\|\mathbf{a}(0)\| < \delta$, then the resulting motion, $\mathbf{a}(t)$, satisfies $\|\mathbf{a}(t)\| < \varepsilon$ for $t > 0$.

Asymptotic Stability

$$\frac{d}{dt}\mathbf{a}(t) = \mathbf{g}(\mathbf{a}(t), \mathbf{p}(t), t)$$

Asymptotic Stability:

The origin is an asymptotically stable equilibrium point if there exists a number $\delta > 0$ such that if $\|\mathbf{a}(0)\| < \delta$, then the resulting motion, $\mathbf{a}(t)$, satisfies $\|\mathbf{a}(t)\| \rightarrow 0$ as $t \rightarrow \infty$.



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Definite Functions

Positive Definite:

A scalar function $V(\mathbf{a})$ is positive definite if $V(\mathbf{0}) = 0$ and $V(\mathbf{a}) > 0$ for $\mathbf{a} \neq 0$.

Positive Semidefinite:

A scalar function $V(\mathbf{a})$ is positive semidefinite if $V(\mathbf{0}) = 0$ and $V(\mathbf{a}) \geq 0$ for all $\mathbf{a}$.

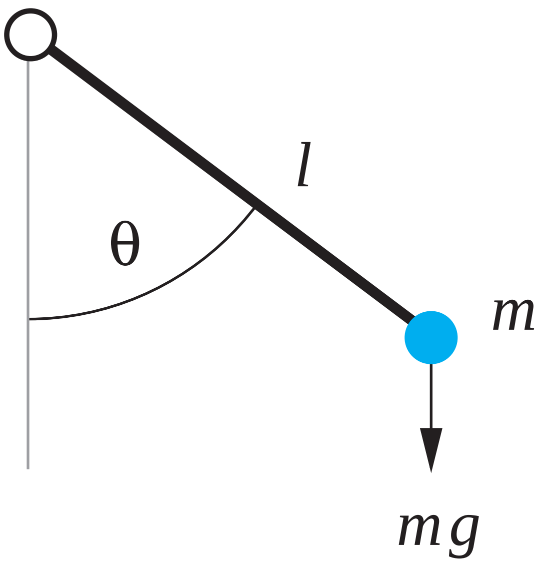
Lyapunov Stability Theorem

$$\frac{d\mathbf{a}}{dt} = \mathbf{g}(\mathbf{a})$$

Theorem 1: Lyapunov Stability Theorem

If a positive definite function $V(\mathbf{a})$ can be found such that $dV(\mathbf{a})/dt$ is negative semidefinite, then the origin $(\mathbf{a} = \mathbf{0})$ is stable for the above system. If a positive definite function $V(\mathbf{a})$ can be found such that $dV(\mathbf{a})/dt$ is negative definite, then the origin $(\mathbf{a} = \mathbf{0})$ is asymptotically stable. In each case, $V(\mathbf{a})$ is called a Lyapunov function of the system.

Pendulum Example



$$ml\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + mg\sin(\theta) = 0$$

State Variable Model

$$a_1 = \theta \qquad \frac{da_1}{dt} = a_2$$

$$a_2 = \frac{d\theta}{dt} \qquad \frac{da_2}{dt} = -\frac{g}{l}\sin(a_1) - \frac{c}{ml}a_2$$

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Equilibrium Point

Check: $\mathbf{a} = \mathbf{0}$

$$\frac{da_1}{dt} = a_2 = 0$$

$$\frac{da_2}{dt} = -\frac{g}{l}\sin(a_1) - \frac{c}{ml}a_2 = -\frac{g}{l}\sin(0) - \frac{c}{ml}(0) = 0$$

Therefore the origin is an equilibrium point.

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Lyapunov Function (Energy)

$$V(\mathbf{a}) = \frac{1}{2}ml^2(a_2)^2 + mgl(1 - \cos(a_1)) \quad \text{(Positive Definite)}$$

$\frac{1}{2}ml^2(a_2)^2$: Kinetic Energy

$mgl(1 - \cos(a_1))$: Potential Energy

Check the derivative of the Lyapunov function:

$$\frac{d}{dt}V(\mathbf{a}) = [\nabla V(\mathbf{a})]^T g(\mathbf{a}) = \frac{\partial V}{\partial a_1}\left(\frac{da_1}{dt}\right) + \frac{\partial V}{\partial a_2}\left(\frac{da_2}{dt}\right)$$

$$\frac{d}{dt}V(\mathbf{a}) = (mgl\sin(a_1))a_2 + (ml^2 a_2)\left(-\frac{g}{l}\sin(a_1) - \frac{c}{ml}a_2\right)$$

$$\frac{d}{dt}V(\mathbf{a}) = -cl(a_2)^2 \le 0$$

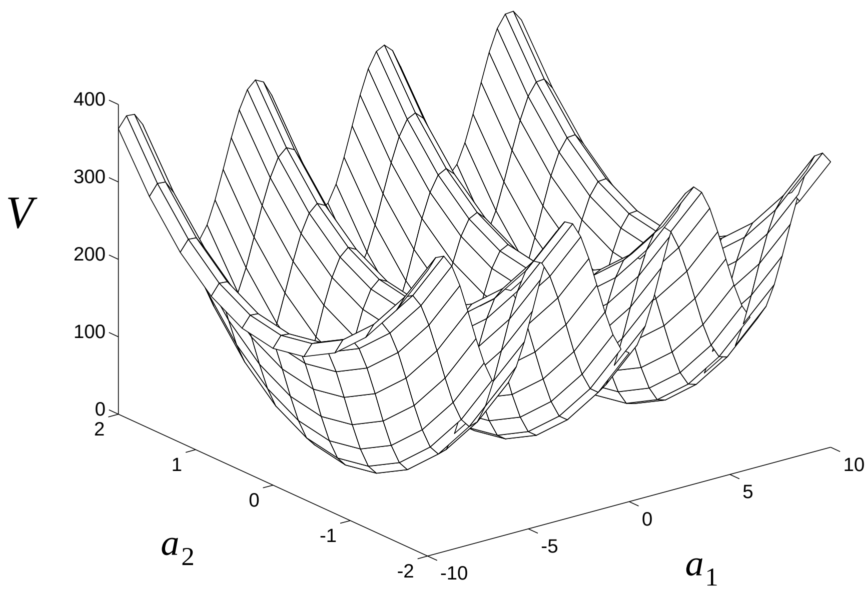
The derivative is negative semidefinite, which proves that the origin is stable in the sense of Lyapunov (at least).

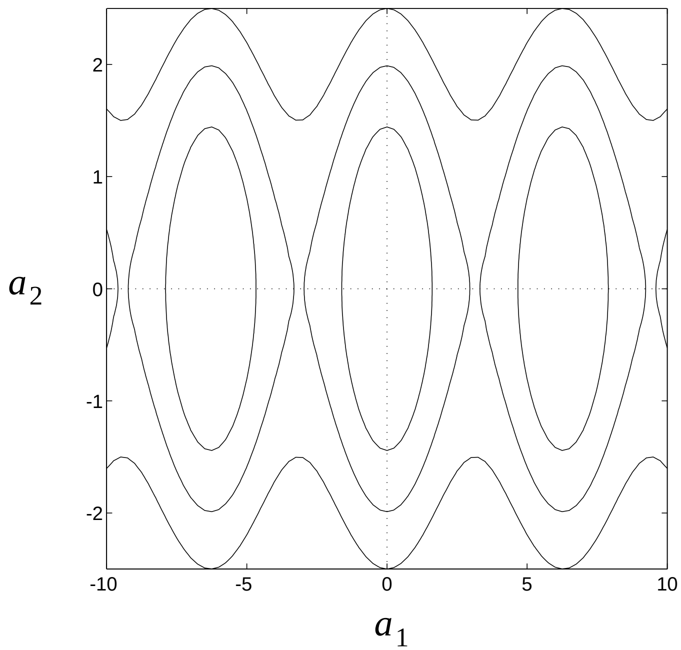
17 Numerical Example

$$g = 9.8, \quad m = 1, \quad l = 9.8, \quad c = 1.96$$

$$\frac{da_1}{dt} = a_2 \qquad \frac{da_2}{dt} = -\sin(a_1) - 0.2a_2$$

$$V = (9.8)^2\left[\frac{1}{2}(a_2)^2 + (1 - \cos(a_1))\right] \qquad \frac{dV}{dt} = -(19.208)(a_2)^2$$



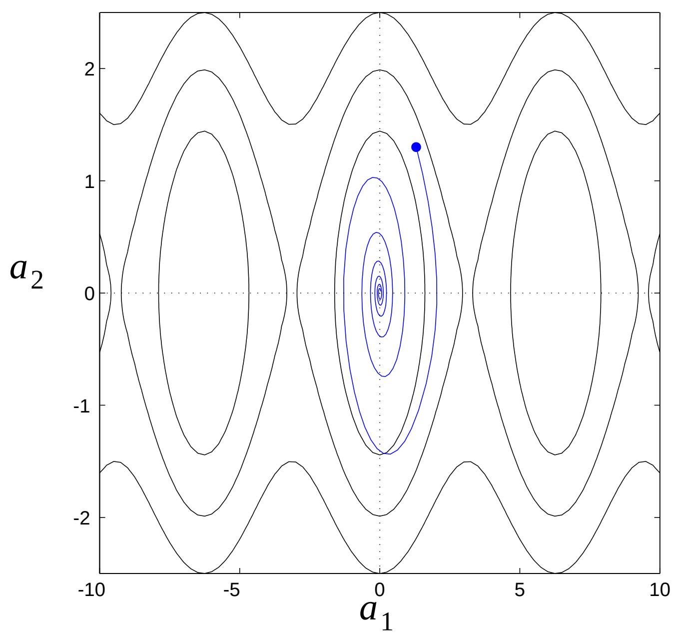


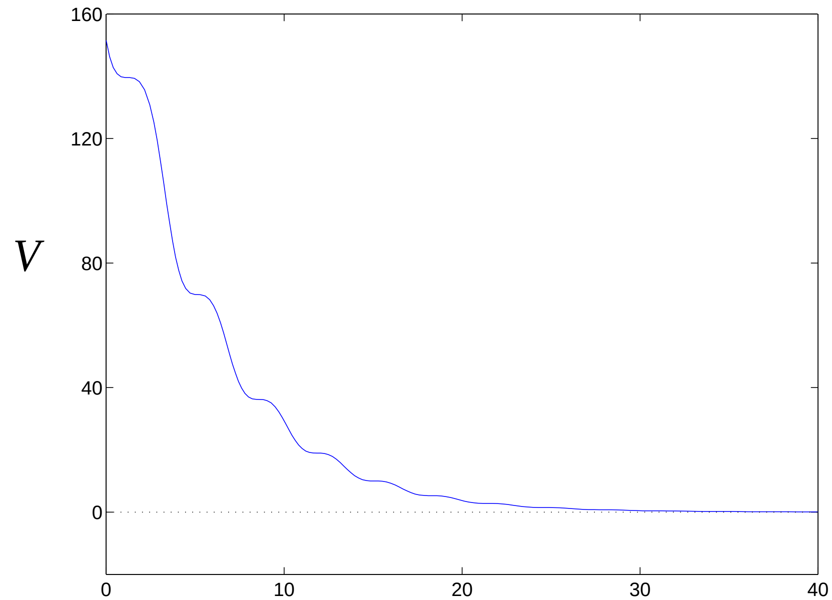
17 Pendulum Response

a_1

$$\mathbf{a}(0) = \begin{bmatrix} 1.3 \\ 1.3 \end{bmatrix}$$

a_2





17 Definitions (Lasalle's Theorem)

Lyapunov Function

Let $V(\mathbf{a})$ be a continuously differentiable function from $\Re^n$ to $\Re$. If G is any subset of $\Re^n$, we say that V is a Lyapunov function on G for the system $d\mathbf{a}/dt = \mathbf{g}(\mathbf{a})$ if

$$\frac{dV(\mathbf{a})}{dt} = (\nabla V(\mathbf{a}))^T \mathbf{g}(\mathbf{a})$$

does not change sign on G.

Set Z

$$Z = \{\mathbf{a}: dV(\mathbf{a})/dt = 0,\ \mathbf{a} \text{ in the closure of } G\}$$

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Definitions

Invariant Set

A set of points in $\Re^n$ is invariant with respect to $d\mathbf{a}/dt = \mathbf{g}(\mathbf{a})$ if every solution of $d\mathbf{a}/dt = \mathbf{g}(\mathbf{a})$ starting in that set remains in the set for all time.

Set L

L is defined as the largest invariant set in Z.

Lasalle's Invariance Theorem

Theorem 2: Lasalle's Invariance Theorem

If V is a Lyapunov function on G for $d\mathbf{a}/dt = \mathbf{g}(\mathbf{a})$, then each solution $\mathbf{a}(t)$ that remains in G for all $t > 0$ approaches $L^{\circ} = L \cup \{\infty\}$ as $t \to \infty$. (G is a basin of attraction for L, which has all of the stable points.) If all trajectories are bounded, then $\mathbf{a}(t) \to L$ as $t \to \infty$.

Corollary 1: Lasalle's Corollary

Let G be a component (one connected subset) of

$$\Omega_{\eta} = \{\mathbf{a}: V(\mathbf{a}) < \eta\}.$$

Assume that G is bounded, $dV(\mathbf{a})/dt \leq 0$ on the set G, and let the set $L^{\circ} = \text{closure}(L \cup G)$ be a subset of G. Then L° is an attractor, and G is in its region of attraction.

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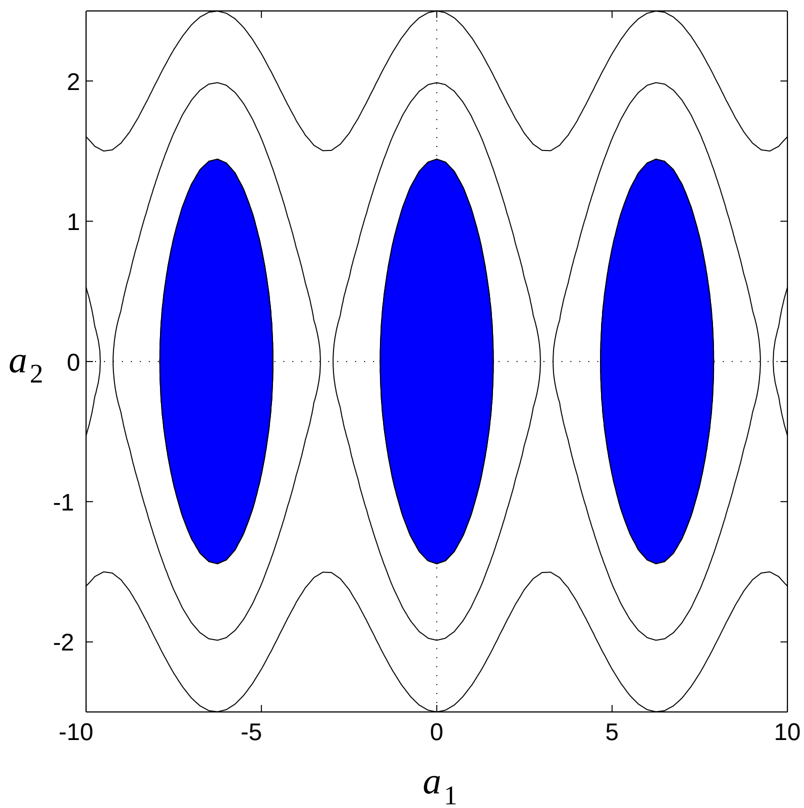
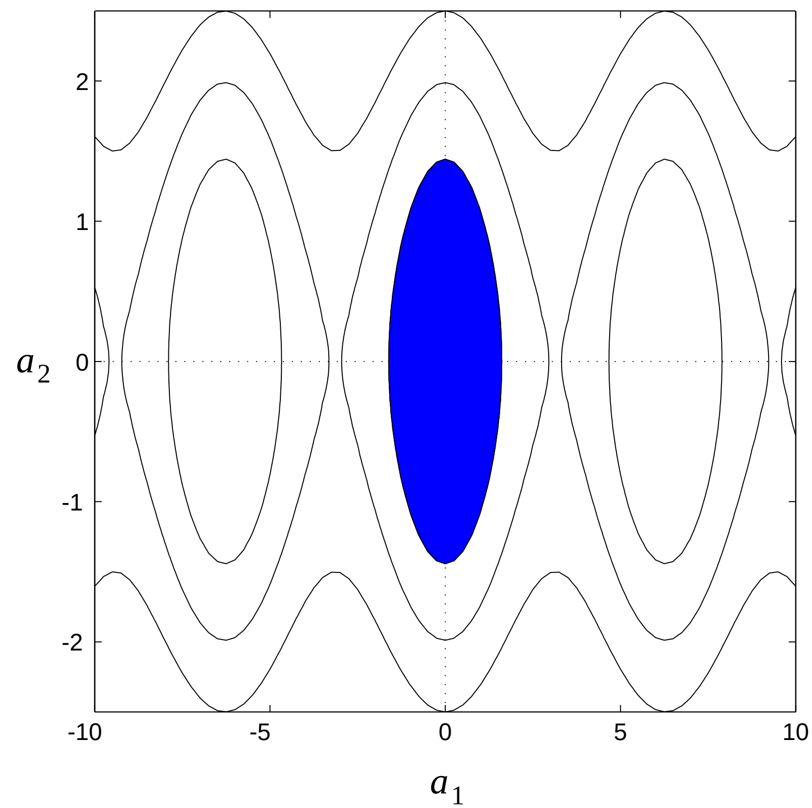
Pendulum Example

$\Omega_{100} = \{\mathbf{a}: V(\mathbf{a}) \leq 100\}$

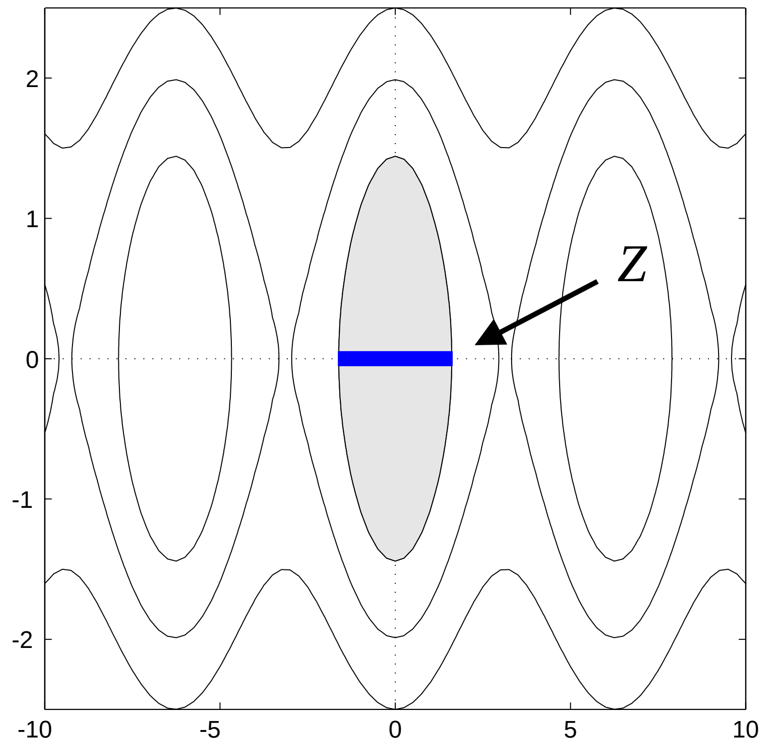


G = One component of Ω_{100}.



Invariant and Attractor Sets

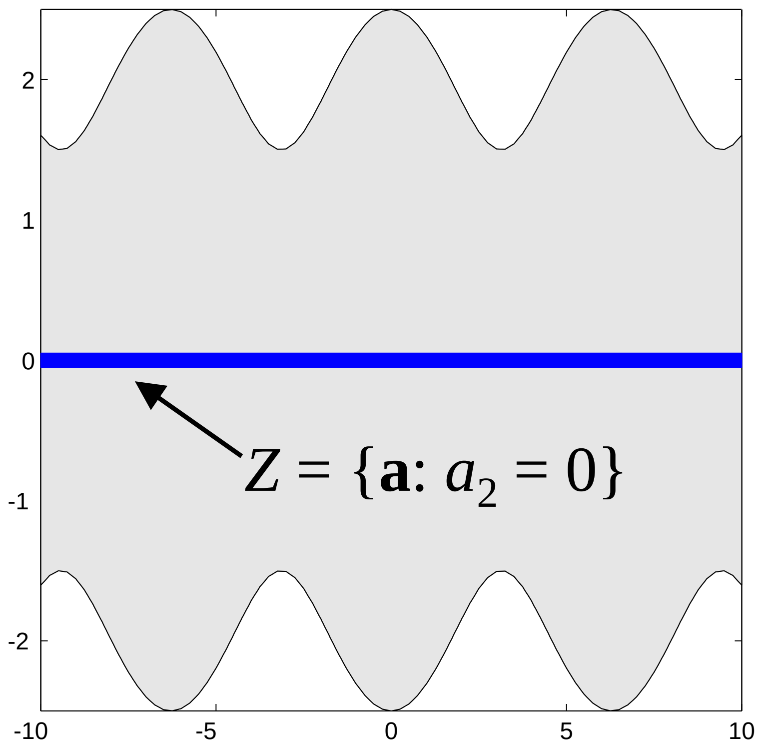
$Z = \{\mathbf{a}: dV(\mathbf{a})/dt = 0,\ \mathbf{a} \text{ in the closure of } G\} = \{\mathbf{a}: a_2 = 0,\ \mathbf{a} \text{ in the closure of } G\}$



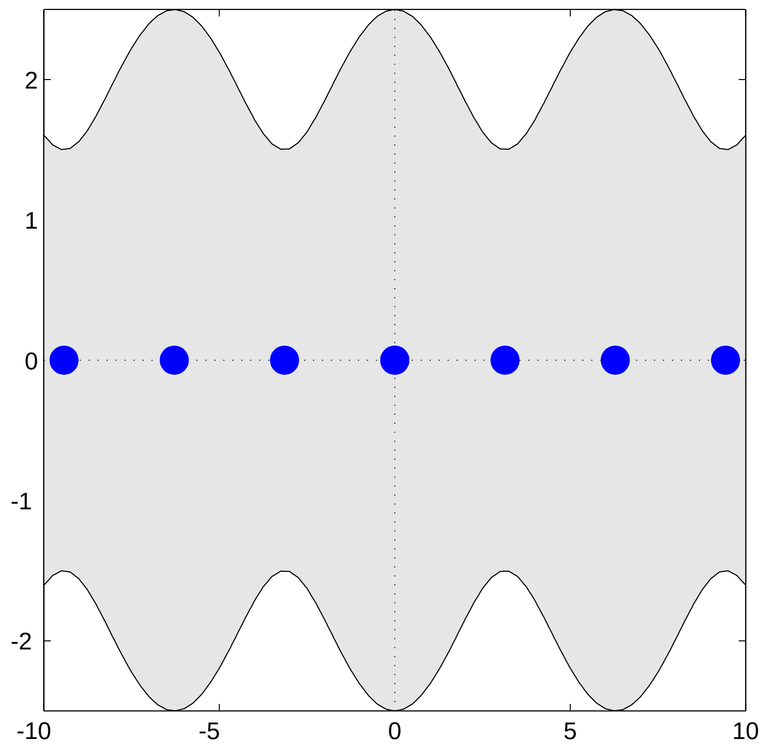
$L = \{\mathbf{a}: \mathbf{a} = 0\}$

Larger G Set

$G = \Omega_{300} = \{\mathbf{a}: V(\mathbf{a}) \le 300\}$



$L^\circ = L = \{\mathbf{a}: a_1 = \pm n\pi,\ a_2 = 0\}$



For this choice of G we can say little about where the trajectorywill converge.

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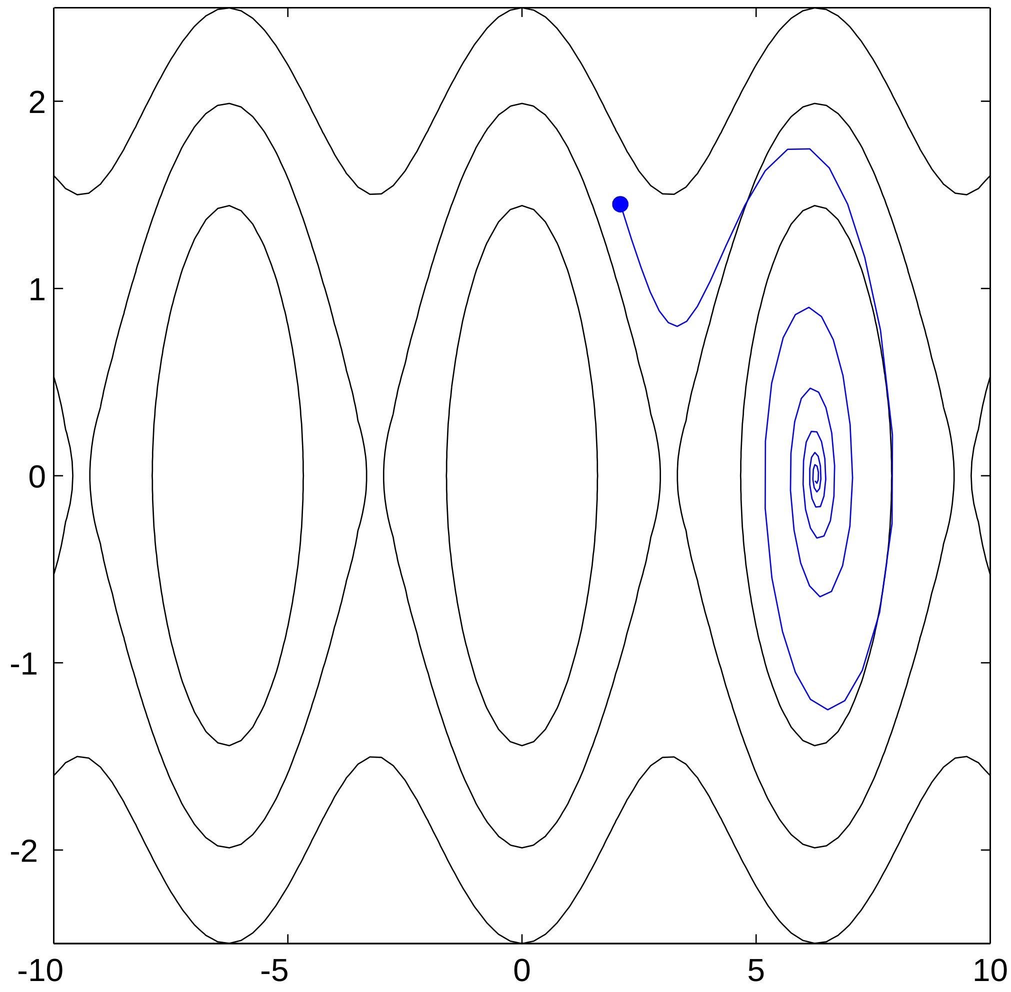
Pendulum Trajectory



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Comments

We want G to be as large as possible, because that will indicate the region of attraction. However, we want to choose V so that the set Z, which will contain the attractor set, is as small as possible.

$V = 0$ is a Lyapunov function for all of $\Re^n$, but it gives no information since $Z = \Re^n$.

If V_1 and V_2 are Lyapunov functions on G, and dV_1/dt and dV_2/dt have the same sign, then $V_1 + V_2$ is also a Lyapunov function, and $Z = Z_1 \cap Z_2$. If Z is smaller than Z_1 or Z_2, then V is a "better" Lyapunov function than either V_1 or V_2. V is always at least as good as either V_1 or V_2.