Learning

How might a neural network learn?

Hebb's Postulate, 1949

The Organization of Behavior

A NEUROPSYCHOLOGICAL THEORY

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London · CHAPMAN & HALL, *Limited*

Hebb's Postulate

A NEUROPHYSIOLOGICAL POSTULATE

Let us assume then that the persistence or repetition of a reverberatory activity (or "trace") tends to induce lasting cellular changes that add to its stability. The assumption ° can be precisely stated as follows: *When an axon of cell A is near enough to excite a cell B and repeatedly or persistently takes part in firing it, some growth process or metabolic change takes place in one or both cells such that A's efficiency, as one of the cells firing B, is increased.*

Hebb's Postulate

When an axon of cell A is near enough to excite a cell B and repeatedly or persistently takes part in firing it,

some growth process or metabolic change takes place in one or both cells

such that A's efficiency, as one of the cells firing B, is increased.

Hebb Restated (Levitan and Kaczmarek)

"When a postsynaptic neuron becomes depolarized [fires], it generates a biochemical reaction or a trophic factor that stabilizes [strengthens] the excitatory synapses that are firing at that time."

Colloquial Hebb

Neurons that fire together wire together.

Levitan and Kaczmarek

"An important aspect of [Hebb's] hypothesis is that a given presynaptic input to a cell need not, *by itself*, be of sufficient strength to induce a large depolarization in its target.

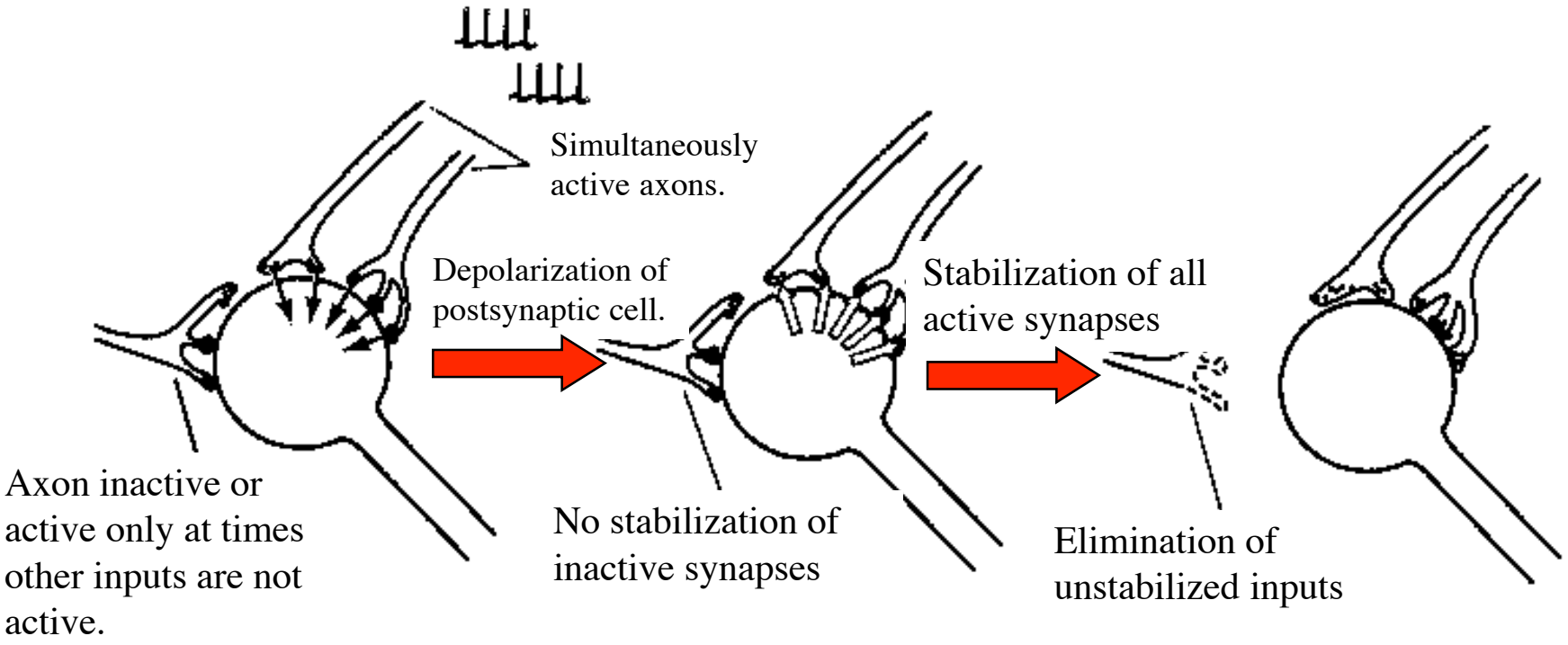
If that input is fired at the same time as a number of other inputs, and their combined action depolarizes the cell, *all* of these inputs will tend to be stabilized."

Levitan and Kaczmarek (cont'd)

- "If, in contrast, a given input fires asynchronously with most of the other inputs onto that cell, this input will tend to be eliminated."

- [This could be called "anti-Hebbian" learning.]

Levitan and Kaczmarek (cont'd)



Hebb's rule. Excitatory synapses that successfully stimulate a post-synaptic neuron, or are active when the postsynaptic neuron is depolarized, are selectively stabilized.

General 1-Perceptron Training

- Assume we are dealing with a **classification** problem (is a given input vector in a set of interest or not?).
- Assuming weights exist for the given set, how to find them?
  - Analytic? (technically not “training”)
    A conventional perceptron doesn’t lend well to analytic solution, due to the discontinuous nature of the function.
  - Successive approximations?

A Somewhat General Approach

- Use a set of **training samples**:
  - Input vectors, each paired with **desired** output
- Choose a network structure.
- Initialize the weights arbitrarily.
- Repeat:
  - Choose a sample
  - Test the network against the sample
  - If answer is incorrect, **adjust** weights
- until all samples test correctly.

Omitted Details

- How to choose a sample?
- How to adjust the weights?

Sample Choice

- Method 1: Simply cycle through all of the samples in a fixed order.
- Method 2: Choose samples randomly, but hopefully with some assurance that each will be checked eventually.

Weight Adjustment

- The Perceptron learning rule (Rosenblatt):
  - If the perceptron gives the correct answer, do nothing.
  - If the perceptron gives the wrong answer, adjust the weights and threshold “in the right direction”, so that it eventually gives the right answer.

When does a Perceptron give the wrong answer?

- Correct answers:

  $w_1 x_1 + w_2 x_2 + \ldots + w_n x_n > \theta$ when $(x_1, x_2, \ldots, x_n)$ is in the set;

  $w_1 x_1 + w_2 x_2 + \ldots + w_n x_n \leq \theta$ when $(x_1, x_2, \ldots, x_n)$ is *not* in the set

When does a Perceptron give the wrong answer?

- *Wrong* answers:

  $w_1 x_1 + w_2 x_2 + \ldots + w_n x_n > \theta$ when $(x_1, x_2, \ldots, x_n)$ is *not* in the set;

  $w_1 x_1 + w_2 x_2 + \ldots + w_n x_n \leq \theta$ when $(x_1, x_2, \ldots, x_n)$ is in the set

Desired Output Value

- Let

  $d(x_1, x_2, \ldots, x_n) =$

  1 if $(x_1, x_2, \ldots, x_n)$ is in the set,

  0 otherwise.

Actual Output Value

- Let

  $a(x_1, x_2, \ldots, x_n) =$

  1 if $w_1 x_1 + w_2 x_2 + \ldots + w_n x_n > \theta$

  0 otherwise.

- Note that a is an implied function of the weights.

Error Value

- We can capture correct vs. incorrect answers succinctly by introducing an *error value* ε:
- $\varepsilon = d(x_1, x_2, \ldots, x_n) - a(x_1, x_2, \ldots, x_n)$
- So that
  - $\varepsilon = 0$ when the correct answer is given
  - $\varepsilon = 1$ when $w_1 x_1 + w_2 x_2 + \ldots + w_n x_n \leq \theta$ but $(x_1, x_2, \ldots, x_n)$ is in the set
  - $\varepsilon = -1$ when $w_1 x_1 + w_2 x_2 + \ldots + w_n x_n > \theta$ but $(x_1, x_2, \ldots, x_n)$ is *not* in the set

Simplification: Threshold conversion

- The threshold can be treated as if one of the weights by introducing a "phantom" input of -1:

$$w_1 x_1 + w_2 x_2 + \ldots + w_n x_n > \theta$$

iff

$$w_1 x_1 + w_2 x_2 + \ldots + w_n x_n - \theta > 0$$

iff

$$w_0 x_0 + w_1 x_1 + w_2 x_2 + \ldots + w_n x_n > 0$$

where w_0 is **defined** to be θ and $x_0 = -1$.

Perceptron training (continued)

- Wrong answer of the first type ($\varepsilon = 1$):
  $w_0 x_0 + w_1 x_1 + w_2 x_2 + \ldots + w_n x_n \leq 0$ when $(x_1, x_2, \ldots, x_n)$ is in the set

- i.e., $w_0 x_0 + w_1 x_1 + w_2 x_2 + \ldots + w_n x_n$ is **too low**.

- To correct for this, we need to **make the sum higher**.

Perceptron training (continued)

- Wrong answer of second type ($\varepsilon = -1$):
  $w_0 x_0 + w_1 x_1 + w_2 x_2 + \ldots + w_n x_n > 0$ when $(x_1, x_2, \ldots, x_n)$ is *not* in the set

- i.e., $w_0 x_0 + w_1 x_1 + w_2 x_2 + \ldots + w_n x_n$ is **too high**.

- To correct for this, we need to make the sum **lower**.

Perceptron training (continued)

- Make $w_0 x_0 + w_1 x_1 + w_2 x_2 + \ldots + w_n x_n$ lower or higher by adjusting weights.
  - $\varepsilon = 1$: Make each contribution $w_i x_i$ **higher**
  - $\varepsilon = -1$: Make each contribution $w_i x_i$ **lower**

- In either case, can ***add*** some multiple η of εx_i to w_i to get the desired effect.

Perceptron training (continued)

- *Add* $\eta\varepsilon x_i$ from w_i to get the desired effect:

$$(w_i + \eta\,\varepsilon x_i)\,x_i = (w_i\,x_i + \eta\varepsilon\,x_i^2)$$
$$> w_i\,x_i \qquad \text{if } \varepsilon > 0$$
$$< w_i\,x_i \qquad \text{if } \varepsilon < 0$$

- η is called the “learning rate”.

Learning Rate η

- η governs the rate at which the training rule converges toward the correct solution.
- Typically $\eta \leq 1$.
- Too small an η produces slow convergence.
- Too large of an η can cause oscillations in the process.

Example

- Train a perceptron to classify according to:
  - (4, 5) yes
  - (6, 1) yes
  - (4, 1) no
  - (1, 2) no
- There will be three weights (w_0, w_1, w_2) where is the w_0 threshold, corresponding to phantom input -1.
- Start with “random” weights, say (0, +1, -1)
- Choose $\eta = 1$.

Perceptron Training Example
One Pass over Data Samples

Fill out this table sequentially:

weights	input	desired	actual	error	new weights
(0, 1, -1)	(-1, 4, 5)	yes			
	(-1, 6, 1)	yes			
	(-1, 4, 1)	no			
	(-1, 1, 2)	no			

Perceptron Training Example
One Pass over Data Samples

weights	input	desired	actual	error	new weights
(0, 1, -1)	(-1, 4, 5)	yes	no	1	(-1, 5, 4)
(-1, 5, 4)	(-1, 6, 1)	yes	yes	0	no change
(-1, 5, 4)	(-1, 4, 1)	no	yes	-1	(0, 1, 3)
(0, 1, 3)	(-1, 1, 2)	no	yes	-1	(1, 0, 1)

Perceptron Training Example
Second Pass over Data Samples

weights	input	desired	actual	error	new weights
(1, 0, 1)	(-1, 4, 5)	yes			
	(-1, 6, 1)	yes			
	(-1, 4, 1)	no			
	(-1, 1, 2)	no			

Perceptron Training Example
Second Pass over Data Samples

weights	input	desired	actual	error	new weights
(1, 0, 1)	(-1, 4, 5)	yes	yes	0	no change
(1, 0, 1)	(-1, 6, 1)	yes	? no	1	(0, 6, 2)
(0, 6, 2)	(-1, 4, 1)	no	yes	-1	(1, 2, 1)
(1, 2, 1)	(-1, 1, 2)	no	yes	-1	(2, 1, -1)

Perceptron Training Example
Third Epoch

weights	input	desired	actual	error	new weights
(2, 1, -1)	(-1, 4, 5)	yes	no	1	(1, 5, 4)
(1, 5, 4)	(-1, 6, 1)	yes	yes	0	no change
(1, 5, 4)	(-1, 4, 1)	no	yes	-1	(2, 1, 3)
(2, 1, 3)	(-1, 1, 2)	no	yes	-1	(3, 0, 1)

Perceptron Training Example
Epoch 4

weights	input	desired	actual	error	new weights
(3, 0, 1)	(-1, 4, 5)	yes	yes	0	no change
(3, 0, 1)	(-1, 6, 1)	yes	no	1	(2, 6, 2)
(2, 6, 2)	(-1, 4, 1)	no	yes	-1	(3, 2, 1)
(3, 2, 1)	(-1, 1, 2)	no	yes	-1	(4, 1, -1)

Perceptron Training Example
Epoch 5

weights	input	desired	actual	error	new weights
(4, 1, -1)	(-1, 4, 5)	yes	no	1	(3, 5, 4)
(3, 5, 4)	(-1, 6, 1)	yes	yes	0	no change
(3, 5, 4)	(-1, 4, 1)	no	yes	-1	(4, 1, 3)
(4, 1, 3)	(-1, 1, 2)	no	yes	-1	(5, 0, 1)

Perceptron Training Example
Epoch 6

weights	input	desired	actual	error	new weights
(5, 0, 1)	(-1, 4, 5)	yes	no	1	(4, 4, 6)
(4, 4, 6)	(-1, 6, 1)	yes	yes	0	no change
(4, 4, 6)	(-1, 4, 1)	no	yes	-1	(5, 0, 5)
(5, 0, 5)	(-1, 1, 2)	no	yes	-1	(6, -1, 3)

Perceptron Training Example
Epoch 7

weights	input	desired	actual	error	new weights
(6, -1, 3)	(-1, 4, 5)	yes	yes	0	no change
(6, -1, 3)	(-1, 6, 1)	yes	no	1	(5, 5, 4)
(5, 5, 4)	(-1, 4, 1)	no	yes	-1	(6, 1, 3)
(6, 1, 3)	(-1, 1, 2)	no	yes	-1	(7, 0, 1)

Perceptron Training Example
Epoch 8

weights	input	desired	actual	error	new weights
(7, 0, 1)	(-1, 4, 5)	yes	no	1	(6, 4, 6)
(6, 4, 6)	(-1, 6, 1)	yes	yes	0	no change
(6, 4, 6)	(-1, 4, 1)	no	yes	-1	(7, 0, 5)
(7, 0, 5)	(-1, 1, 2)	no	yes	-1	(8, -1, 3)

Perceptron Training Example
Epoch 9

weights	input	desired	actual	error	new weights
(8, -1, 3)	(-1, 4, 5)	yes	yes	0	no change
(8, -1, 3)	(-1, 6, 1)	yes	no	1	(7, 5, 2)
(7, 5, 2)	(-1, 4, 1)	no	yes	-1	(8, 1, 3)
(8, 1, 3)	(-1, 1, 2)	no	no	0	(8, 1, 3)

Perceptron Training Example
Epoch 10

weights	input	desired	actual	error	new weights
(8, 1, 3)	(-1, 4, 5)	yes	yes	0	no change
(8, 1, 3)	(-1, 6, 1)	yes	yes	0	no change
(8, 1, 3)	(-1, 4, 1)	no	no	0	no change
(8, 1, 3)	(-1, 1, 2)	no	no	0	no change

Perceptron Training Example Conclusion

- A perceptron with weights (8, 1, 3) correctly classifies all inputs.
- The "yes" criterion is therefore:
  $-8 + x_1 + 3\,x_2 > 0$ [i.e. $x_1 + 3\,x_2 > 8$]
- Check:

input x1, x2	desired	actual
(4, 5)	yes	yes
(6, 1)	yes	yes
(4, 1)	no	no
(1, 2)	no	no

Perceptron Training Algorithm (1)

- Inputs:
  - A list of training samples, each of the form
    - $[d(x_1, x_2, \ldots, x_n), -1, x_1, x_2, \ldots, x_n]$
    - (d is the desired output, -1 the phantom input)
  - An initial weight vector $[w_0, w_1, w_2, \ldots, w_n]$
    - (w_0 is the threshold)
  - A learning rate η

Perceptron Training Algorithm (2)

- Outputs:
  - If the set of samples is linearly separable, a vector of weights $[w_0, w_1, w_2, \ldots, w_n]$ such that with these weights the perception properly separates the training samples.
  - If the set of samples is not linearly separable, then the algorithm diverges.

Perceptron Training Algorithm (3)

- Operation:
  - Set $[w_0, w_1, w_2, \ldots, w_n]$ = initial weights;
  - **while**(there is a sample not correctly classified)
    - Let $[d(x_1, x_2, \ldots, x_n), -1, x_1, x_2, \ldots, x_n]$ be an incorrectly classified sample.
    - Let $\varepsilon = d(x_1, x_2, \ldots, x_n) - a(x_1, x_2, \ldots, x_n)$, where $a(x_1, x_2, \ldots, x_n) = (\Sigma\, w_i x_i > 0)$.
    - Vector-add to $[w_0, w_1, w_2, \ldots, w_n]$ the vector

$$\Delta w = \varepsilon\, \eta\, [-1, x_1, x_2, \ldots, x_n]$$

Perceptron learning rule

Perceptron Training Algorithm Modifications for practical usage

- Put a ***limit*** on the number of iterations, so that the algorithm will terminate even if the sample set is not linearly separable.

- Include an ***error bound*** as an extra input. The algorithm can stop as soon as the portion of mis-classified samples is less than this bound (as opposed to requiring perfect classification, which would be an error bound of 0).

- Generate the initial weights ***randomly***, so that the user does not have to specify them.

Uncertainties about the perceptron training algorithm

- If there is no separating hyperplane, the perceptron will never classify the samples 100% correctly.
- But there is nothing to stop it from trying, unless we add a limit on the number of steps.
- Will the algorithm always find acceptable weights if there is a separating hyperplane?

Termination of the Perceptron Algorithm

- Clearly, if the algorithm terminates, then a separating hyperplane has been found (in the form of the weight vector).
- How to show that it will terminate in the case that there is a separating hyperplane?

Algorithm Termination in General

- A common way to show an algorithm terminates is to find a quantity that provably cannot increase forever.
- Identification of such a quantity in this case is not so obvious.

Perceptron Algorithm Termination

- From experience with the algorithm, we can observe that the sum of the squares of the weight vector tends to increase.

- We can try to get a bound on the rate of increase.

Perceptron Convergence Theorem (1)

If the sample set is linearly separable,
then the perceptron training algorithm will always converge, even if η is fixed at 1.

- Proof: Suppose that the sample set is linearly separable. Then let w* be a weight vector that separates the samples.

- We can assume ||w*|| = 1, since if w* separates, then so does w*/ ||w*|| and the latter vector has length 1.

Perceptron Convergence Theorem (2)

- We can assume, without loss of generality, that all samples are in the "yes" set, for if not, we can replace that sample with its negative

  $(\Sigma\, w_i\, x_i < 0)$ iff $(\Sigma\, -w_i\, x_i > 0)$.

- Let $x^1, x^2, x^3, \ldots, x^k$ be the *mis*-classified samples in a training sequence, i.e. the ones that are used to adjust the weights.

- Let w^k be the resulting weight vector, with w^0 as the initial weight vector.

Perceptron Convergence Theorem (3)

- Since the samples are all in the “yes” set, the error ε = 1 always (as opposed to -1), so

$$w^k = w^0 + x^1 + x^2 + x^3 + \ldots + x^k$$

- Multiplying both sides by w*:

$$w^* w^k = w^* w^0 + w^* x^1 + w^* x^2 + w^* x^3 + \ldots + w^* x^k$$

the products being inner vector products.

Perceptron Convergence Theorem (4)

- Since the x^k are all in the "yes" set, each inner product on the right is > 0 (because w* is a correct classifier):

$$w^*w^k = w^*w^0 + w^*x^1 + w^*x^2 + w^*x^3 + \ldots + w^*x^k$$

- So there is a value $\delta > 0$ such that for each i:

$$w^*w^i > \delta$$

which gives

$$w^*w^k > w^*w^0 + k\delta$$

Perceptron Convergence Theorem (5)

- Now look at the ***squares*** of the $\|w^k\|$.
- The Cauchy-Schwarz inequality tells us

$$\|w^*\|^2 \, \|w^k\|^2 > (w^* w^k)^2$$

and since $\|w^*\|^2 = 1$, we have

$$\|w^k\|^2 > (w^* w^k)^2.$$

- Substituting for $w^* w^k$, where $w^* w^k > w^* w^0 + k\delta$, we get

$$\|w^k\|^2 > (w^* w^0 + k\delta)^2$$

i.e. the sequence of terms $\|w^k\|^2$ grows
at least quadratically with k.

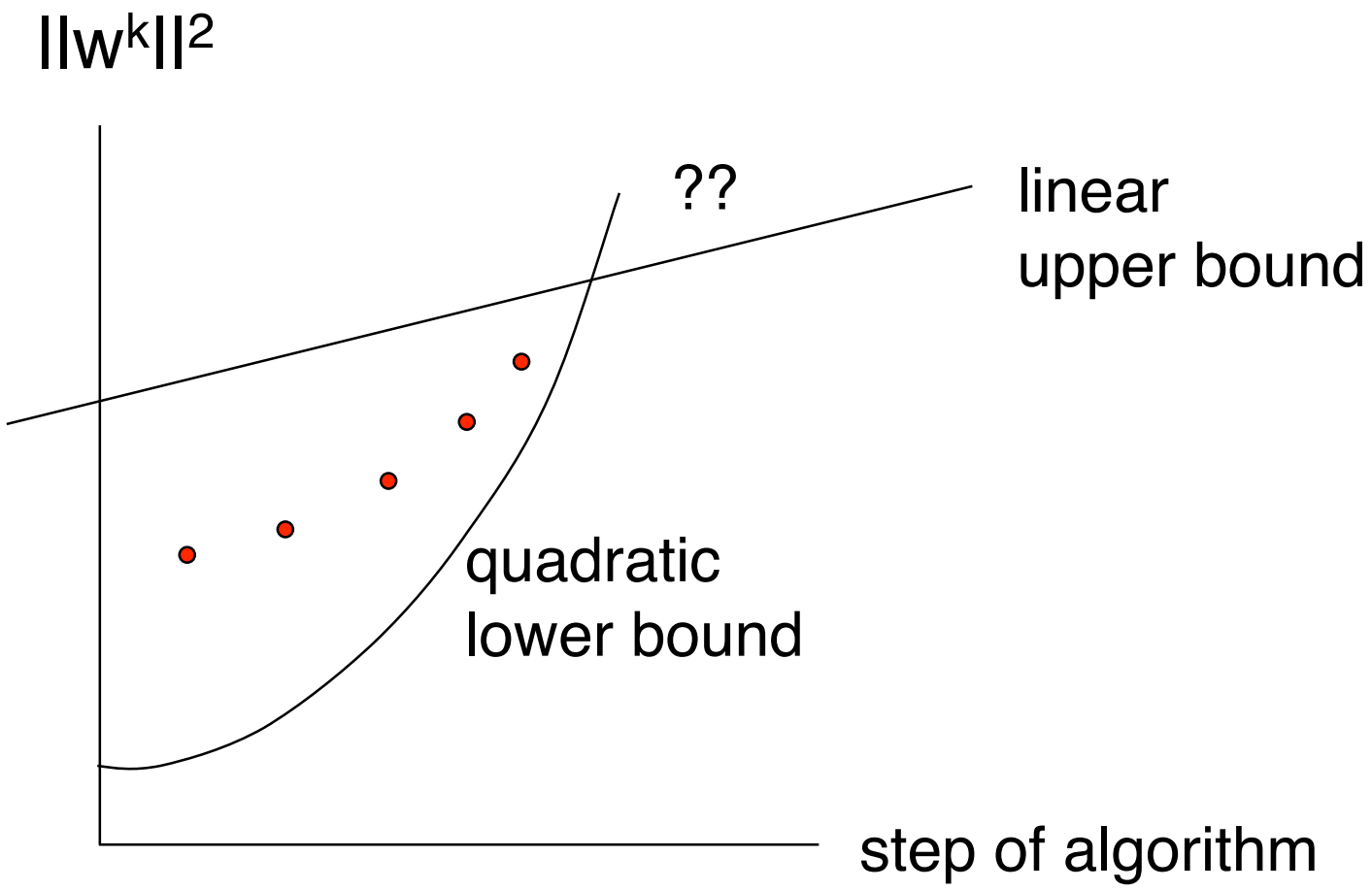
Perceptron Convergence Theorem (6)

- The sequence of terms $\|w^k\|^2$ grows ***at least quadratically*** with k.

- Pivotal idea:

  If we can now show that the *same* sequence grows ***at most linearly***, we will have established that the sequence must be finite, it cannot go on forever.

Perceptron Convergence Theorem (7)



Perceptron Convergence Theorem (8)

- Since $w^k = w^{k-1} + x^k$,
  $\|w^k\|^2 = w^k \, w^k$
  $= w^{k-1} \, w^{k-1} + 2 \, w^{k-1} \, x^k + x^k \, x^k$
- But since the x^k are misclassified and in the "yes" category, $w^{k-1} \, x^k < 0$.
  Therefore
  $\|w^k\|^2 < \|w^{k-1}\|^2 + \|x^k\|^2$
- We telescope-sum these k inequalities to get
  $\|w^k\|^2 < \|w^0\|^2 + k \max_k \|x^k\|^2$

  constant,
  since the set of samples is finite.

  which is our *linear* upper bound on $\|w^k\|^2$.

Handling Multiple Outputs

- Occasionally problems will have more than two distinct output values.
- In order to fit a perceptron to such a task, the designer must map the set of output values into **vectors** of 0's and 1's.
- This can be accomplished in many ways, but using a "one-hot" encoding is probably the safest, although not the cheapest in terms of neurons.

Issue of equality in linear separability

- We stated the decision criterion for the Perceptron in terms of strict inequalities < and >. But as we know, it might happen that the sum of the weights = the threshold.

- For a given set of samples, I claim that if there is set of weights that separates with equality allowed, there is also a set that separates without using =.
- Intuitively, we only need to “nudge” the separating hyperplane so that no sample points lie on it.

- Therefore no generality has been lost in the assumption of strict inequality.
- Practically, the = can be resolved by grouping it with either < or > consistently.

Types of Data

- **Numerical**: Ordering makes sense
- **Categorical**: No natural ordering
- To use other than some kind of bit-vector encoding for categorical data is to suggest that there is a numerical ordering when there really is none.
- Choosing an arbitrary ordering is not likely to work for training perceptrons, except in special cases.

Bit-Vector Implementation

- A bit-vector encoding can be implemented by training a separate perceptron for each bit of the encoding.

- In effect, the weight vector becomes a weight matrix, with one row per perceptron and one column per input, as in the one-perceptron case.

Weight Vector/Matrix

- $[w_0, w_1, w_2, \ldots, w_n]$ 1 perceptron

- $\begin{pmatrix} w_{10}, & w_{11}, w_{12}, & \ldots, w_{1n} \\ w_{20}, & w_{21}, w_{22}, & \ldots, w_{2n} \\ & \ldots & \\ w_{m0}, & w_{m1}, w_{m2}, & \ldots, w_{mn} \end{pmatrix}$ m perceptrons

Some Unhappiness About Perceptron Training

- When a perceptron gives the right answer, no learning takes place.
- Anything below the threshold is interpreted as “no”, even if it *just* below the threshold.
- Might it be better to train the neuron based on ***how far*** below the threshold it is?
- This idea is developed in the **Adaline** training algorithm.