Parametric cubic curves

- Interpolating
- Hermitian
- Catmull-Rom
- Bezier
- B-spline

Hermitian splines

- *Specify endpoint position*
- *Specify derivative at endpoint*

Hermitian

- $X(t) = at^3 + bt^2 + ct + d$
- $X(0) = 3$, $X(1) = 2$
- $X'(0) = 1$, $X'(1) = 0$
- Write 4 equations that determine the coefficients a, b, c, and d.

Hermitian: Constraints

- X(0):
- X(1):
- X'(0):
- X'(1):

Hermitian: Constraints

- X(0): $d = 3$
- X(1): $a+b+c+d = 2$
- X'(0): $c = 1$
- X'(1): $3a+2b+c=0$

Hermitian Matrix Form

$$\begin{bmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} - \\ - \\ - \\ - \end{bmatrix}$$

Hermitian Matrix Form

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

find X(t) for this example

Hint

$$\begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix} = ?$$

X(t)

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 1 \\ 3 \end{pmatrix}$$

$X(t) = 3t^3 - 5t^2 + t + 3$

Verify

$X(t) = 3t^3 - 5t^2 + t + 3$

- $X(0) = 3,\ X(1) = 2$
- $X'(0) = 1,\ X'(1) = 0$

General Solution: X(t)

$x(t) = at^3 + bt^2 + ct + d$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X(0) \\ X(1) \\ X'(0) \\ X'(1) \end{pmatrix}$$

General Solution: Y(t)

$Y(t) = at^3 + bt^2 + ct + d$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Y(0) \\ Y(1) \\ Y'(0) \\ Y'(1) \end{pmatrix}$$

General Solution: Z(t)

$Z(t) = at^3 + bt^2 + ct + d$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Z(0) \\ Z(1) \\ Z'(0) \\ Z'(1) \end{pmatrix}$$

Hermitian Basis Matrix

$$\begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Exercise

- $X(0) = 3$, $X(1) = 2$, $X'(0) = 1$, $X'(1) = 0$
- $Y(0) = 2$, $Y(1) = 2$, $Y'(0) = 0$, $Y'(1) = 1$

- Write the equations
- Plot the curve for t in [0,1]

Equations

- $X(0) = 3$, $X(1) = 2$, $X'(0) = 1$, $X'(1) = 0$
- $Y(0) = 2$, $Y(1) = 2$, $X'(0) = 0$, $X'(1) = 1$

- Equations:
- $X(t) = 3t^3 - 5t^2 + t + 3$
- $Y(t) = t^3 - t^2 + 2$

Hermitian Description

- Basis Matrix
- Basis (blending) Functions

Hermitian: X(t)

$$X(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

General Matrix: X

$$X(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X(0) \\ X(1) \\ X'(0) \\ X'(1) \end{bmatrix}$$

Blending Functions: X

$$X(t) = \begin{bmatrix} P_1(t) & P_2(t) & P_3(t) & P_4(t) \end{bmatrix} \begin{bmatrix} X(0) \\ X(1) \\ X'(0) \\ X'(1) \end{bmatrix}$$

Hermitian Blending Functions

$$\begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 2t^3-3t^2+1, & -2t^3+3t^2, & t^3-2t^2+t, & t^3-t^2 \end{bmatrix}$$

$P_1(t)$ $P_2(t)$ $P_3(t)$ $P_4(t)$

Do it yourself demo

Hermitian Curves

Hermitian: problem 1

Specifying derivatives is awkward,

particularly when many curves are connected with derivative continuity.

Parametric Continuity

C^i: The 0th, 1st, 2nd, …, ith derivative of adjacent curves agree at their boundary points.

Geometric Continuity

$G^0 = C^0$

For i>0, G^i means

- G^0 continuity plus
- The 1st, 2nd, …, ith derivative of adjacent curves are proportional at boundary point.

Do it yourself demo

Continuity

Exercise

- First Hermitian curve x(t):
  - $x_1(0)$, $x_1(1)$, $x'_1(0)$, $x'_1(1)$
- Second Hermitian curve:
  - What conditions provide C^i & G^i continuity for i=0,1?

Hermitian: G^0

$x_2(0)=x_1(1)$

Hermitian: G^1

G^1: $x_2(0)=x_1(1)$ and $x_2'(0)=\alpha x_1'(1)$ for some α

(note: same α factor applies to y(t) and z(t))

Hermitian: C^0

$x_2(0)=x_1(1)$

Hermitian: C^1

$x_2(0)=x_1(1)$ and $x_2'(0)=x_1'(1)$

64

Hermitian problem

Not invariant under affine transformations

In other words: transforming (rotate, scale, translate) the control values does not yield the transformed curve

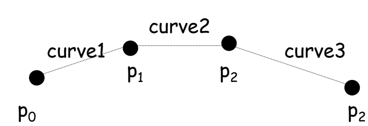
65

Parametric cubic curves

- Interpolating
- Hermitian
- **Catmull-Rom** enforces C^1 continuity
- Bezier
- B-spline

66

Catmull-Rom Spline: C^1



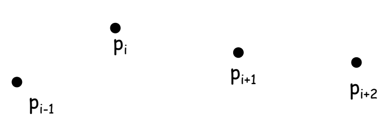
tangent at p_1 = (1/2) $\langle p_2-p_0 \rangle$

tangent at p_2 = (1/2) $\langle p_3-p_1 \rangle$

67

Catmull-Rom Basis Matrix

- Compute the basis matrix for the Catmull-Rom spline from p_i to p_{i+1}



68

Catmull-Rom constraints

$X(t) = at^3 + bt^2 + ct + d$

assume $p_i=(x_i,y_i)$

- $X(0)=d=x_i$
- $X(1)=a+b+c=x_{i+1}$
- $X'(0)=c=(x_{i+1}-x_{i-1})/2$
- $X'(1)=3a+2b+c=(x_{i+2}-x_i)/2$

69

Catmull-Rom Basis Matrix

0	0	0	1
1	1	1	0
0	0	1	0
3	2	1	0

a
b
c
d

=

0	1	0	0
0	0	1	0
-.5	0	.5	0
0	-.5	0	.5

x_{i-1}
x_i
x_{i+1}
x_{i+2}

Catmull-Rom Basis Matrix

a
b
c
d

=0.5*

-1	3	-3	1
2	-5	4	-1
-1	0	1	0
0	2	0	0

x_{i-1}
x_i
x_{i+1}
x_{i+2}

Catmull-Rom problem

Using control points to specify point on curve
and tangent at adjacent point can be awkward.