Algorithms
Computer Science 140 & Mathematics 168
Instructor: B. Thom
Spring 2004
Homework 11b
Due on 4/13/04

1. [**10 points**] **Investigating Ones Cuts (Yum)!**

   Suppose someone gave you an original graph G and its residual G_r, and that G_r had no path from s to t. Suggest how you would modify BFS or DFS to:

   (a) Return sets S and T.

   (b) Return the set of edges in the minimum capacity cut.

   Either pseudo-code or a description in English is fine. Keep your writeup brief. We're looking for the big-idea here, not the crossed t's and dotted i's.

2. [**25 points**] **Space Shuttle Profit Optimization!**

   After successful careers at Millisoft, Hurts Car Rental, and the brokerage firm of Weil, Proffet, and Howe, you've been hired by NASA to help optimize their Space Shuttle program.

   On a given mission, NASA will consider a set of experiments that industrial sponsors would like to conduct (for example, "Does gravity affect Spam?"). Let $E = \{E_1, E_2, \ldots, E_m\}$ denote the set of experiments under consideration. Let p_j denote the amount of money the sponsor will pay to conduct experiment E_j. The experiments use a set $I = \{I_1, I_2, \ldots, I_n\}$ of instruments to conduct the experiments. For each experiment E_j, let R_j be the subset of I that contains all of the instruments needed to conduct experiment E_j. Notice that this allows a single instrument to be used in multiple experiments. The instruments are potentially large and heavy and the cost of taking instrument I_k is c_k dollars. Your job is to determine which experiments should be performed in order to maximize the net revenue, which is the total income from the performed experiments minus the total cost of all instruments carried. Amazingly, this problem can be solved using **network flow**!

   We'll construct a network flow problem as follows: The network contains a source vertex, s, vertices $I_1, I_2, \ldots, I_n$, vertices $E_1, E_2, \ldots, E_m$, and a sink vertex t. For each instrument I_k there is a directed edge from s to vertex I_k with capacity c_k (the cost of taking this instrument). For each experiment E_j there is an edge from vertex E_j to t with capacity p_j (the payment for this experiment). Finally, if instrument I_k is in set R_j (meaning that instrument I_k is needed for experiment E_j) then there is a directed edge from vertex I_k to vertex E_j with **infinite** capacity.

   (a) First, try this. Consider a situation in which there are three experiments E_1, E_2, E_3 which will bring in 10, 6, and 6 dollars, respectively. Also there are four instruments I_1, I_2, I_3, and I_4 which cost 3, 2, 5, and 7 dollars, respectively to take on the shuttle. Experiment E_1 requires instruments I_1 and I_2, experiment E_2 requires instruments I_1 and I_3, and experiment E_3 requires instruments I_3 and I_4. Using brute-force, enumerate all seven possible combinations of experiments that could be taken and determine which combination is the most profitable. What is this combination and what is the net revenue?

(b) For the example problem above, construct the corresponding network flow problem. Find the maximum flow in the network (show your residual graphs at each step) and then find the corresponding cut whose capacity is equal to this flow.

(c) Let's let S denote the vertices that are on the same side of the above cut as vertex s and let T denote the vertices that are on the same side of the cut as t. What do you notice about the instruments and experiments that are in the set T?

(d) Now, let τ be the sum of the payments that NASA would receive for taking all of the possible experiments. In this case, $\tau = 10 + 6 + 6 = 22$. From τ, subtract the capacity of the cut you found above. Surprise! What is this number and how does it appear to relate to this problem?

(e) Finally, we're ready to generalize all of this into an efficient algorithm for solving the profit maximization problem in general. Assume that we've set up a network flow problem corresponding to a given set of experiments and instruments. Prove that for any cut with finite total capacity, if an experiment E_j is in T (the side of the cut containing vertex t), then all of the instruments used in this experiment must **also** be in T. Be careful and precise.

(f) Now prove that the maximum net revenue that can be achieved is simply the total sum τ of the payments that would be received for taking all of the experiments minus the capacity of the cut found by a network-flow algorithm.

(g) Now summarize all of this step-by-step, producing a general purpose algorithm for:

i. Finding the maximum net revenue, given a set of experiments, instruments, and the corresponding payments and costs.

ii. Specifying which experiments should be run to obtain this revenue.

Derive the worst-case running time of the algorithm assuming there are m experiments and n instruments (use these variables in your final run-time analysis instead of V and E).

3. [**20 Points**] **FUHLishness!**

The Harvey Mudd **F**aculty **U**nicycle **H**ockey **L**eague (FUHL) has four teams numbered 1 through 4. Each team has lost a certain number of games so far, as shown in Table 1 below, and has a certain number of games remaining against the other teams in the league, as shown in Table 2.

Aside: for those of you that are latex-ing your solutions, you can download the eps file that contains these tables via:

`http://www.cs.hmc.edu/courses/2004/spring/cs140/homework/nw_fuhl.eps`

Team number	Losses so far
1	1
2	4
3	2
4	10

Table 1

Remaining schedule

Team number	4	3	2
3	4		
2	5	4	
1	7	4	2

Table 2

The league champion is the team with the minimum number of losses at the end of the season. We wish to know if Team 4 has any hope of being the league champion, in spite of the fact that it has lost so many games already.

(a) Show how this problem can be cast as a network flow problem by constructing the corresponding flow network.

(b) Now solve the network flow problem. For each iteration of the algorithm, show the residual graph and the flow in the network so far.

(c) Use the maximum flow to determine a scenario (which teams win and lose) in which Team 4 is the league champion.

4. **[25 points] And Yet More Network Reliability!**

Give an efficient algorithm for the *s,t-connectivity problem* and argue that it is correct. The *s,t-connectivity problem* is defined as follows: given a directed or undirected graph $G = (V, E)$ and elements $s, t \in V$, $s \neq t$, decide if there exists k edge-disjoint paths from s to t and if so, find them.