CS 141, Advanced Topics in Algorithms
Spring 2004
Homework 1a
Due Monday, January 26

- Please recall that solutions to homework assignments in this class must be typeset, preferably in LaTeX. You should keep electronic versions of all of your homework submissions.
- Please also recall that homeworks are due at the very beginning of class.

1. **[15 Points] Stirling Numbers of the Second Kind!** In class we used "Stirling Numbers of the Second Kind" to help us express regular powers as the sum of falling powers. We defined Stirling's Triangle and showed how it was constructed. Specifically, let $\left\{ \begin{array}{c} n \\ k \end{array} \right\}$ denote the Stirling number in the k^{th} column of row n of Stirling's Triangle. In class, we said that the rule for building Stirling's Triangle works like this:

   (a) $\left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\} = 1$. (That is, the element at the top of the triangle is 1.)

   (b) $\left\{ \begin{array}{c} n \\ 0 \end{array} \right\} = 0$ for all $n \geq 1$. (That is, the left edge of the triangle is all 0's.)

   (c) $\left\{ \begin{array}{c} n \\ n \end{array} \right\} = 1$ for all $n \geq 1$. (That is, the right edge of the triangle is all 1's.)

   (d) $\left\{ \begin{array}{c} n \\ k \end{array} \right\} = \left\{ \begin{array}{c} n-1 \\ k-1 \end{array} \right\} + k \left\{ \begin{array}{c} n-1 \\ k \end{array} \right\}$ for any $n \geq 1$.

   This is very similar to the binomial coefficients $\binom{n}{k}$ in Pascal's Triangle. The Binomial coefficients have some special significance: $\binom{n}{k}$ is the number of ways to choose k objects from n distinct objects where order does not matter. Do the Stirling Numbers of the Second Kind have some meaning as well (other than being useful in Discrete Calculus)? Yes! It turns out that $\left\{ \begin{array}{c} n \\ k \end{array} \right\}$ counts the number of different ways to partition n distinct objects into k nonempty sets. For example consider partitioning the three objects $1, 2, 3$ into two sets, neither of which is empty. There are only three ways to do this:

   - $\{1, 2\}$, $\{3\}$.
   - $\{1, 3\}$, $\{2\}$.
   - $\{2, 3\}$, $\{1\}$.

   Notice that $\left\{ \begin{array}{c} 3 \\ 2 \end{array} \right\} = 3$ (just look at Stirling's Triangle!).

(a) Given that $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ counts the number of ways to partition n distinct objects into k nonempty sets, give a combinatorial argument that explains why $\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\} + k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}$. (Show that both sides count the same thing in two different ways.)

(b) Give a combinatorial argument to explain why $\left\{ \begin{matrix} n \\ 2 \end{matrix} \right\} = 2^{n-1} - 1$.

(c) Give a combinatorial argument to explain why $\left\{ \begin{matrix} n \\ n-1 \end{matrix} \right\} = \binom{n}{2}$.

2. **[20 Points] Stirling Numbers and The Discrete Calculus.** In the previous problem you gave a combinatorial proof that:

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\} + k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}$$

From this identity, we could build the Stirling Triangle of the Second Kind. Then, Ran made the following wild claim in class:

$$x^n = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^{\underline{k}}$$

for any integer $n \geq 0$. This identity was very useful in computing all sorts of summations using the Discrete Calculus. Now, you will prove this identity.

(a) First, show that $x \cdot x^{\underline{k}} = x^{\underline{k+1}} + kx^{\underline{k}}$.

(b) Now, use induction on n to show that

$$x^n = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^{\underline{k}}$$

The identity that you proved in part (a) may be useful to you.

3. **[10 Points] Deriving Amazing Formulae with Discrete Calculus!** Use Discrete Calculus to derive a closed-form formula for

$$\sum_{k=0}^{n} k^3.$$

Show your work in detail.