CS 141
Advanced Topics in Algorithms
Spring 2004
Problem Set 6b
Due Monday, March 1

1. [**10 Points**] **Just Review.** Recall that the *Matroid Theorem* states the following: "The Matroid Greedy Algorithm returns a basis of maximum weight." Write down the proof of this theorem in your own words. Be precise and rigorous.

2. [**40 Points**] **The Millisoft Matching Problem!** In this problem we investigate another discrete optimization problem and its corresponding matroid.

   Gill Bates has come up with a revolutionary new scheme for matching up pen pals. He plans to offer the service through his MilliSoft Network web site in the near future.

   Assume that there are n subscribers to the service. Ideally, every subscriber in the network will be paired up with exactly one other subscriber. For compatibility reasons, only certain pairs of subscribers can be matched and it therefore may not be possible to find a match for every subscriber. Each subscriber stipulates a single fee that they are willing to pay to get matched. Millisoft would like to find a matching that maximizes the total fees that it can collect from the subscribers.

   This optimization problem can be modelled as a graph in which vertices correspond to subscribers, edges correspond to subscribers that can potentially be matched, and the weight on each vertex represents the fee that will be paid if that vertex is matched. Recall that a *matching* in a graph is defined to be a set of edges of G such that no two edges share a common vertex. Our objective is to find a matching that maximizes the sum of the weights on the matched vertices.

   Notice that this graph matching problem is slightly different from the one that arose in the task assignment problem. First, the graph that models this problem is not necessarily bipartite. In addition, in this problem every vertex has an associated weight.

   To solve this problem we consider a new matroid called a *matching matroid*. Let $G = (V, E)$ be a graph. Define $X \subseteq V$ to be *matchable* if there exists some matching $M \subseteq E$ such that every vertex in X is incident on some edge in M. For convenience, we say that an edge in M *covers* its endpoints or M covers the set X. Unlike the case of the bipartite matchings in the task assignment problem, an edge in matching M may be incident on zero, one, or two vertices in X. Let $\mathcal{I}$ be the set of all matchable subsets of V. Assume that we already have an algorithm that determines whether or not a set is matchable and, if it is matchable, the algorithm constructs a matching for it.

   (a) If $M_M = (V, \mathcal{I})$ is a matroid and w is a weight function from the vertices to the positive reals, show that a basis of maximum weight in M_M corresponds to an optimal solution to this optimization problem.

(b) We will now show that M_M is a matroid in a sequence of steps. Begin by showing that M_M satisfies the heredity property.

(c) To show that the exchange property is satisfied, begin by considering $X, Y \in \mathcal{I}$ such that $|X| > |Y|$ and consider matchings M_X and M_Y that cover X and Y, respectively. Argue that if M_Y covers a vertex in $X - Y$ then the exchange property is satisfied.

(d) Now consider the case that M_Y covers no vertex in $X - Y$. Color the edges in $M_X - M_Y$ black, color the edges in $M_Y - M_X$ white, and color the edges in $M_X \cap M_Y$ gray. Show that the gray edges cover at least as many vertices in Y as in X.

(e) Recall that an *alternating path* and an *alternating cycle* is a path and cycle, respectively, that alternates between black and white edges. Show that the edges on the alternating cycles cover at least as many vertices in Y as in X.

(f) Show that *collectively* all the alternating paths contain more vertices of X than Y. Note that this implies that some alternating path contains more vertices in X than Y.

(g) Show that an endpoint of an alternating path cannot be in both X and Y.

(h) Show that there exists an alternating path $v_1, \ldots, v_k$ such that $v_1 \in X - Y$ and $v_k \notin Y$.

(i) Show that $Y + v_1 \in \mathcal{I}$ by showing that a matching exists that covers this set. Conclude that M_M is a matroid.

3. **[10 Points] Try it Out!** As an example of the optimization problem described in Problem 1, consider a network comprising five subscribers, a, b, c, d, e, represented by the graph in the figure below. An edge between two vertices indicates that the corresponding subscribers are compatible and may be matched to one another. The number next to each vertex represents the fee (or weight) associated with that subscriber. Use the Matroid Greedy Algorithm to find an optimal solution for this instance of the optimization problem. Briefly explain what the Matroid Greedy Algorithm selects at each iteration. You will have to do the matchability test by inspection.