CS 81 Assignment 2 for Wed., Feb. 2 Solutions

Exercises in Hein book:

1. Ex. 10 on p 383

Proof of $(A \rightarrow B) \wedge (\neg A \rightarrow C) \vdash (A \wedge B) \vee (\neg A \wedge C)$:

1.	$(A \rightarrow B) \wedge (\neg A \rightarrow C)$	Premise
2.	$(A \rightarrow B)$	$\wedge e_1$
3.	$(\neg A \rightarrow C)$	$\wedge e_2$
4.	$(A \vee \neg A)$	LEM
5.	A	Assumption
6.	B	$\rightarrow e$ 5, 2
7.	$A \wedge B$	$\wedge i$ 5, 6
8.	$(A \wedge B) \vee (\neg A \wedge C)$	$\vee i_1$ 7
9.	$\neg A$	Assumption
10.	C	$\rightarrow e$ 9, 3
11.	$\neg A \wedge C$	$\wedge i$ 9, 10
12.	$(A \wedge B) \vee (\neg A \wedge C)$	$\vee i_1$ 11
13.	$(A \wedge B) \vee (\neg A \wedge C)$	$\vee e$ 4, 5-8, 9-12

Proof of $(A \wedge B) \vee (\neg A \wedge C) \vdash (A \rightarrow B) \wedge (\neg A \rightarrow C)$:

1.	$(A \wedge B) \vee (\neg A \wedge C)$	Premise
2.	$A \wedge B$	Assumption
3.	A	Assumption
4.	B	$\wedge e_2$ 2
5.	$A \rightarrow B$	$\rightarrow i$ 3-4
6.	$\neg A$	Assumption
7.	A	$\wedge e_1$ 2
8.	$\bot$	$\neg e$ 7,6
9.	C	$\bot e$ 8
10.	$\neg A \rightarrow C$	$\rightarrow i$ 6-9
11.	$(A \rightarrow B) \wedge (\neg A \rightarrow C)$	$\wedge i$ 5, 10
12.	$\neg A \wedge C$	Assumption
13.	A	Assumption
14.	$\neg A$	$\wedge e_1$ 12
15.	$\bot$	$\neg e$ 13, 14
16.	B	$\bot e$ 15
17.	$A \rightarrow B$	$\rightarrow i$ 13-16
18.	$\neg A$	Assumption
19.	C	$\wedge e_2$ 12
20.	$\neg A \rightarrow C$	$\rightarrow i$ 18-19
21.	$(A \rightarrow B) \wedge (\neg A \rightarrow C)$	$\wedge i$ 17, 20
22.	$(A \rightarrow B) \wedge (\neg A \rightarrow C)$	$\vee e$ 1, 2-11, 12-21

2. Ex. 8.d on p 367

Use Quine's method to show that this is a tautology:

$(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

Substitute for A:

A = F: $(F \to (B \to C)) \to ((F \to B) \to (F \to C))$
which simplifies to $T \to (T \to T)$
which evaluates to T

A = T: $(T \to (B \to C)) \to ((T \to B) \to (T \to C))$
which simplifies to $(B \to C) \to (B \to C)$
which evaluates to T regardless of the value of $(B \to C)$.

3. Ex. 8.f on p 367

Use Quine's method to show that this is a tautology:

$(A \to B) \to ((C \vee A) \to (C \vee B))$

Substitute for C:

C = F: $(A \to B) \to ((F \vee A) \to (F \vee B))$
which simplifies to $(A \to B) \to (A \to B)$
which evaluates to T regardless of the value of $(A \to B)$.

C = T: $(A \to B) \to ((T \vee A) \to (T \vee B))$
which simplifies to $(A \to B) \to (T \to T)$
which simplifies to $(A \to B) \to T$
which simplifies to T

4. Which of these are tautologies?

a. $((\neg p) \to p)$ **Not a tautology.** Assignment p = F induces F.

b. $p \to ((\neg p) \to p)$ **A tautology.** Both p = F and p = T induce T.

c. $((\neg p) \to p) \to p$ **A tautology.** Both p = F and p = T induce T.

d. $p \vee (\neg p)$ **A tautology.** Both p = F and p = T induce T.

e. $(p \wedge (\neg p)) \to p$ **A tautology**, since $(p \wedge (\neg p))$ is F for any assignment

f. $(p \vee (\neg p)) \to p$ **Not a tautology**. Assignment p = F induces F.

g. $(p \wedge q) \to (p \vee q)$ **A tautology**. All 4 combinations induce T.

h. $((\neg p) \to q) \to (p \vee q)$ **A tautology**. For p = F, simplifies to $q \to q$ which is a tautology. For p = T, simplifies to $T \to T$ which is a tautology.

i. $(\neg(p \rightarrow q)) \rightarrow ((\neg p) \rightarrow q)$ **A tautology**.
Using Quine's method: q = F simplifies to $\neg\neg p \rightarrow \neg\neg p$, which is T. q = T simplifies to $F \rightarrow T$, which is T.

5. When something is not a tautology, there must be at least one assignment (of truth values to propositions) that induces the value false.
For each non-tautology above, show such an assignment.

Already done in the solution to 4.

6. Logical equivalence is expressed by the connective $\equiv$, where $A \equiv B$ is viewed as an abbreviation for $(A \rightarrow B)$ and $(B \rightarrow A)$.
Which of the following are tautologies? Justify your answers.

j. $(p \rightarrow q) \equiv ((\neg p) \vee q)$
is a tautology. For p = F, simplifies to $T \equiv T$. For p = T, simplifies to $q \equiv q$, both of which are always T.

k. $(\neg(p \wedge q)) \equiv ((\neg p) \wedge (\neg q))$
is not a tautology. Consider the assignment p = F, q = T, which simplifies to $T \equiv F$, which is F.

l. $(\neg(p \vee q)) \equiv ((\neg p) \wedge (\neg q))$
is a tautology. For p = F, simplifies to $\neg q \equiv \neg q$. For p = T, simplifies to $F \equiv F$, both of which are always T.

m. $(p \rightarrow (q \rightarrow (r \rightarrow s))) \equiv ((p \wedge (q \wedge r)) \rightarrow s)$
is a tautology. Using Quine's method, for p = F, simplifies to $T \equiv T$.
For p = T, simplifies to $(q \rightarrow (r \rightarrow s)) \equiv ((q \wedge r) \rightarrow s)$.
For q = F, simplifies to $T \equiv T$.
For q = T, simplifies to $(r \rightarrow s) \equiv (r \rightarrow s)$, both of which are T.

n. $(((p \rightarrow q) \rightarrow r) \rightarrow s) \equiv (((p \wedge q) \wedge r) \rightarrow s)$
is not a tautology. For p = T, q = F, r = F, s = F, becomes $F \equiv T$, which is F.

o. $(p \rightarrow q) \equiv ((\neg q) \rightarrow (\neg p))$
is a tautology. For p = T, simplifies to $q \equiv \neg\neg q$ which is T.
For p = F, simplifies to $T \equiv T$.

7. Devise a natural deduction proof rules for ≡ introduction and elimination.

Since $A \equiv B$ is the same as $(A \rightarrow B) \wedge (B \rightarrow A)$, we can adapt the rules for →:

≡ introduction rule: This rule says that to prove we can do sub proofs of B assuming A and of A assuming B. Since the first of these proves $A \rightarrow B$, while the second proves $B \rightarrow A$, it is clear that rule is justified.

$$\frac{\boxed{\begin{array}{c}A\\ \cdot\\ \cdot\\ \cdot\\ B\end{array}} \qquad \boxed{\begin{array}{c}B\\ \cdot\\ \cdot\\ \cdot\\ A\end{array}}}{A \equiv B} \qquad \equiv i$$

≡ elimination rules: The first rule says that if we have A and also $A \equiv B$, then we can conclude B. This is justified since it is the conclusion of the →e rule, with $A \rightarrow B$ in place of $A \equiv B$. The other rule is like the →e rule, with $B \rightarrow A$ in place of $A \equiv B$.

$$\frac{A \qquad A \equiv B}{B} \qquad \equiv e_1$$

$$\frac{B \qquad A \equiv B}{A} \qquad \equiv e_2$$