CS 81 Assignment 3 Solutions for Wed., Feb. 9

1. Demonstrate how the construction in the proof of the completeness theorem ($\models \eta$ implies $\vdash \eta$) would construct a proof of the tautology

$$(p \rightarrow q) \rightarrow \neg(p \wedge \neg q).$$

There are two proposition symbols p, q. We need to construct the following mini-proofs:

a. $\neg p, \neg q \vdash (p \rightarrow q) \rightarrow \neg(p \wedge \neg q)$
b. $\neg p, \quad q \vdash (p \rightarrow q) \rightarrow \neg(p \wedge \neg q)$
c. $\quad p, \neg q \vdash (p \rightarrow q) \rightarrow \neg(p \wedge \neg q)$
d. $\quad p, \quad q \vdash (p \rightarrow q) \rightarrow \neg(p \wedge \neg q)$

Part a. $\neg p, \neg q \vdash (p \rightarrow q) \rightarrow \neg(p \wedge \neg q)$:

The corresponding assignment is $\alpha(p) = F$, $\alpha(q) = F$.
From the meaning of **$\rightarrow$,** we need a proof of **either**

$\neg p, \neg q \vdash \neg(p \rightarrow q)$

or

$\neg p, \neg q \vdash \neg(p \wedge \neg q)$:

Evaluating:

$\alpha(\neg(p \rightarrow q)) = F$
$\alpha(\neg(p \wedge \neg q)) = T$

so we choose the second, and construct a proof of

$\neg p, \neg q \vdash \neg(p \wedge \neg q)$.

Since $\alpha(p \wedge \neg q) = F$,
we need a proof of either

$\neg p, \neg q \vdash \neg p$

or of

$\neg p, \neg q \vdash \neg(\neg q)$

Evaluating:

$\alpha(\neg p) = T$
$\alpha(\neg\neg q) = F$

So we must choose the former. But that has a proof of length 0, so our mini-proof for this case is:

1.	$\neg p$	Premise
2.	$\neg q$	Premise
3.	$(p \wedge \neg q)$	Assumption
4.	p	$\wedge e_1$ 3
5.	$\bot$	$\neg e$ 4, 1
6.	$\neg(p \wedge \neg q)$	$\neg i$ 3-5
7.	$(p \rightarrow q)$	Assumption
8.	$\neg(p \wedge \neg q)$	copy 6
9.	$(p \rightarrow q) \rightarrow \neg(p \wedge \neg q)$	$\rightarrow i$ 7-8

Part b. $\neg p, q \vdash (p \to q) \to \neg(p \land \neg q)$:

The corresponding assignment is $\alpha(p) = F$, $\alpha(q) = T$,

$\alpha(p \to q) = T$

$\alpha(\neg(p \land \neg q)) = T$

so we need a proof of

$\neg p, \quad q \vdash \neg(p \land \neg q)$

Evaluating,

$\alpha(p \land \neg q) = F$

so we need proofs of either or

$\neg p, \quad q \vdash \neg p$

$\neg p, \quad q \vdash \neg(\neg q)$

The first has length 0. So our mini-proof is:

10.	$\neg p$	Premise
11.	q	Premise
12.	$p \land \neg q$	Assumption
13.	$\neg q$	$\land e_2$ 12
14.	$\perp$	$\neg e$ 11, 13
15.	$\neg(p \land \neg q)$	$\neg i$ 12-14
16.	$p \to q$	Assumption
17.	$\neg(p \land \neg q)$	copy 15
18.	$(p \to q) \to \neg(p \land \neg q)$	$\to i$ 16-17

Part c. $p, \neg q \vdash (p \to q) \to \neg(p \land \neg q)$:

The corresponding assignment is $\alpha(p) = T$, $\alpha(q) = F$,

$\alpha(p \to q) = F$

$\alpha(\neg(p \land \neg q)) = F$

so we need a proof of

$p, \neg q \vdash \neg(p \to q)$

which implies we need proofs of

$p, \neg q \vdash p$ and of $p, \neg q \vdash \neg q$

So our mini-proof is:

19.	p	Premise
20.	$\neg q$	Premise
21.	$(p \to q)$	Assumption
22.	q	$\to e$ 19, 21
23.	$\perp$	$\neg e$ 22. 20
24.	$\neg(p \to q)$	$\neg i$ 21-23
25.	$(p \to q)$	Assumption
26.	$\perp$	$\neg e$ 25, 24
27.	$\neg(p \land \neg q)$	$\perp e$ 26

28.	$(p \to q) \to \neg(p \wedge \neg q)$	$\to$i 25-27

Part d. $p, q \vdash (p \to q) \to \neg(p \wedge \neg q)$:

The corresponding assignment is $\alpha(p) = T$, $\alpha(q) = T$,

$\alpha(p \to q) = T$

$\alpha(\neg(p \wedge \neg q)) = T$

so we need a proof of

$p, q \vdash \neg(p \wedge \neg q)$

but

$\alpha(p \wedge \neg q) = F$

which implies that we need a proof of either

$p, q \vdash \neg p$

or

$p, q \vdash \neg(\neg q)$

Clearly we can get only the second, which requires a proof of

$p, q \vdash q$

which has length 0.

29.	p	Premise
30.	q	Premise
31.	$\neg(\neg q)$	$\neg\neg$i 30
32.	$p \wedge \neg q$	Assumption
33.	$\neg q$	$\wedge e_2$ 32
34.	$\perp$	$\neg$e 30, 33
35.	$\neg(p \wedge \neg q)$	$\neg$i 32-34
36.	$p \to q$	Assumption
37.	$\neg(p \wedge \neg q)$	copy 35
38.	$(p \to q) \to \neg(p \wedge \neg q)$	$\to$i 36-37

Now we “glue together” the essences of the mini-proofs:

1.	p ∨ ¬p	LEM	
2.	¬p	Assumption	
3.	q ∨ ¬q	LEM	
4.	¬q	Assumption	Part a.
5.	p ∧ ¬q	Assumption	
6.	p	$\wedge e_1$ 5	
7.	⊥	¬e 2, 6	
8.	¬(p ∧¬q)	¬i 5-7	
9.	(p → q)	Assumption	
10.	¬(p ∧ ¬q)	copy 8	
11.	(p → q) → ¬(p ∧ ¬q)	→i 9-10	
12.	q	Assumption	Part b.
13.	p∧ ¬q	Assumption	
14.	¬q	$\wedge e_2$ 13	
15.	⊥	¬e 11, 14	
16.	¬(p ∧ ¬q)	¬i 13-15	
17.	p → q	Assumption	
18.	¬(p ∧ ¬q)	copy 16	
19.	(p → q) → ¬(p ∧ ¬q)	→i 17-18	
20.	(p → q) → ¬(p ∧ ¬q)	∨e 3, 12-19, 4-11	
21.	p	Assumption	
22.	q ∨ ¬q	LEM	
23.	¬q	Assumption	Part c.
24.	(p → q)	Assumption	
25.	q	→e 21, 24	
26.	⊥	¬e 25, 23	
27.	¬(p → q)	¬i 24-26	
28.	(p → q)	Assumption	
29.	⊥	¬e 28. 27	
30.	¬(p ∧ ¬q)	⊥e 29	
31.	(p → q) → ¬(p ∧¬q)	→i 28-30	
32.	q	Assumption	Part d.
33.	¬(¬q)	¬¬i 32	
34.	p ∧ ¬q	Assumption	
35.	¬q	$\wedge e_2$ 34	
36.	⊥	¬e 32, 35	
37.	¬(p ∧ ¬q)	¬i 34-36	
38.	p → q	Assumption	
39.	¬(p ∧ ¬q)	copy 37	
40.	(p → q) → ¬(p ∧ ¬q)	→i 38-39	
41.	(p → q) → ¬(p ∧¬q)	∨e 22, 23-31, 32-40	
42.	(p → q) → ¬(p ∧¬q)	∨e 1, 2-20, 21-41	

2. Construct a natural deduction proof for these sequents:

a. $(\forall x)\ p(x) \lor (\forall x)\ q(x) \vdash (\forall x)\ (p(x) \lor q(x))$

1.	$(\forall x)\ p(x) \lor (\forall x)\ q(x)$	Premise
2.	$(\forall x)\ p(x)$	Assumption
3.	x_0	
4.	$p(x_0)$	$\forall$e 2
5.	$p(x_0) \lor q(x_0)$	$\lor i_1$ 4
6.	$(\forall x)\ (p(x) \lor q(x))$	$\forall$i 3-5
7.	$(\forall x)\ q(x)$	Assumption
8.	x_0	
9.	$q(x_0)$	$\forall$e 7
10.	$p(x_0) \lor q(x_0)$	$\lor i_2$ 10
11.	$(\forall x)\ (p(x) \lor q(x))$	$\forall$i 8-11
12.	$(\forall x)\ (p(x) \lor q(x))$	$\lor$e 1, 2-6, 7-11

b. $(\exists x)\ p(x) \lor (\exists x)\ q(x) \vdash (\exists x)\ (p(x) \lor q(x))$

1.	$(\exists x)\ p(x) \lor (\exists x)\ q(x)$	Premise
2.	$(\exists x)\ p(x)$	Assumption
3.	$x_0\ \ p(x_0)$	Assumption
4.	$p(x_0) \lor q(x_0)$	$\lor i_1$ 3
5.	$(\exists x)\ (p(x) \lor q(x))$	$\exists$i 4
6.	$(\exists x)\ (p(x) \lor q(x))$	$\exists$e 2, 3-5
7.	$(\exists x)\ q(x)$	Assumption
8.	$x_0\ \ q(x_0)$	Assumption
9.	$p(x_0) \lor q(x_0)$	$\lor i_2$ 8
10.	$(\exists x)\ (p(x) \lor q(x))$	$\exists$i 9
11.	$(\exists x)\ (p(x) \lor q(x))$	$\exists$e 7, 8-10
12.	$(\exists x)\ (p(x) \lor q(x))$	$\lor$e 1, 2-6, 7-11

3. Construct natural deduction proofs for two analogs of the DeMorgan rules:

   a. $(\forall x)\ p(x) \vdash \neg(\exists x)\ \neg p(x)$

1.	$(\forall x)\ p(x)$	Premise
2.	$(\exists x)\ \neg p(x)$	Assumption
3.	$x_0\ \ \neg p(x_0)$	Assumption
4.	$p(x_0)$	$\forall$e 1
5.	$\perp$	$\neg$e 4, 3
6.	$\perp$	$\exists$e 2, 3-5
7.	$\neg(\exists x)\ \neg p(x)$	$\neg$i 2-6

   b. $(\exists x)\ p(x) \vdash \neg(\forall x)\ \neg p(x)$

1.	$(\exists x)\ p(x)$	Premise
2.	$x_0\ \ p(x_0)$	Assumption
3.	$(\forall x)\ \neg p(x)$	Assumption
4.	$\neg p(x_0)$	$\forall$e 3
5.	$\perp$	$\neg$e 2, 4
6.	$\neg(\forall x)\ \neg p(x)$	$\neg$i 3-5
7.	$\neg(\forall x)\ \neg p(x)$	$\exists$e 1, 2-6

4. Express the premises and conclusion below in predicate logic. Then prove the combination as a sequent:

   a. All computer scientists can program.
   b. Anyone who can program is proficient with logic.
   c. No one who is proficient with logic holds a public office.
   d. Therefore no one holding a public office is a computer scientist.

Let $C(x)$ mean x is a computer scientist.
Let $P(x)$ mean x can program.
Let $L(x)$ mean x is proficient with logic.
Let $H(x)$ mean x holds a public office.

The sequent to be shown is:

$$(\forall x)(C(x) \rightarrow P(x)),\\ (\forall x)(P(x) \rightarrow L(x)),\\ \neg(\exists x)(L(x) \wedge H(x)) \quad \vdash \quad \neg(\exists x)(H(x) \wedge C(x))$$

The proof is on the next page.

1.	$(\forall x)(C(x) \rightarrow P(x))$	Premise
2.	$(\forall x)(P(x) \rightarrow L(x))$	Premise
3.	$\neg(\exists x)(L(x) \wedge H(x))$	Premise
4.	$(\exists x)\ (H(x) \wedge C(x))$	Assumption
5.	$x_0\ (H(x_0) \wedge C(x_0))$	Assumption
6.	$H(x_0)$	$\wedge e_1$ 5
7.	$C(x_0)$	$\wedge e_2$ 5
8.	$C(x_0) \rightarrow P(x_0)$	$\forall e$ 1
9.	$P(x_0)$	$\rightarrow e$ 7, 8
10.	$P(x_0) \rightarrow L(x_0)$	$\forall e$ 2
11.	$L(x_0)$	$\rightarrow e$ 9, 10
12.	$L(x_0) \wedge H(x_0)$	$\wedge i$ 11, 6
13.	$(\exists x)(L(x) \wedge H(x))$	$\exists i$ 12
14.	$\perp$	$\neg e$ 13, 3
15.	$\perp$	$\exists e$ 4, 5-14
16.	$\neg(\exists x)(H(x) \wedge C(x))$	$\neg i$ 4-15

5. Convince yourself that the following assertions can each be true or false for a specific 2-ary predicate p on a non-empty domain:

   a. Reflexive: $(\forall x)\ p(x, x)$

   b. Symmetric: $(\forall x)\ (\forall y)\ (p(x, y) \rightarrow p(y, x))$

   c. Transitive: $(\forall x)\ (\forall y)\ (\forall z)\ (p(x, y) \wedge p(y, z) \rightarrow p(x, z))$

   The following English-language argument is given to justify that Transitive and Symmetric together imply Reflexive: "Let x_0 be an arbitrary element. Suppose that y_0 is such that $p(x_0, y_0)$. Then by the symmetric property, also $p(y_0, x_0)$. But then by the transitive property, from $p(x_0, y_0) \wedge p(y_0, x_0)$ we get $p(x_0, x_0)$. Since x_0 was arbitrary, it follows that $(\forall x)\ p(x, x)$, which is just the reflexive property."

   If this argument is correct, show the proof using natural deduction. If it is incorrect, demonstrate why.

   Answer: The argument is not correct. The statement

   "Suppose that y_0 is such that $p(x_0, y_0)$."

   is not correct. To permit that statement, we must have first established
   $(\exists y)\ p(x_0, y)$
   but this is not necessarily true for p in general.
   For example, on a set {0, 1}, we could have p = {(1, 1)} (but not p(0, y) for any y). This p is symmetric and transitive, but not reflexive.