CS 81 Assignment 4 Solutions for Wed., Feb. 16

1. Give a natural deduction proof of the following, based on the Group Theory axioms given in class:

$$(\forall x)\ (i(i(x)) = x)$$

(Here i is the “inverse” function.)

Here is my plan:

a. $(x' x'') = u$	G3R with x' for x
b. $(x\ (x' x'')) = (x\ u)$	equals for equals, $x = x$ and $(x' x'') = u$ as arguments to the group operator
c. $(x\ u) = x$	G2R
d. $(x\ (x' x'')) = x$	transitivity of = b, c
e. $(x\ (x' x'')) = ((x\ x')\ x'')$	G1 with x' for y, x'' for z
f. $(x\ x') = u$	G3R with x for x
g. $(x\ (x' x'')) = (u\ x'')$	=substitute u for $(x\ x')$ on rhs of e
h. $(u\ x'') = x''$	G2L
i. $(x\ (x' x'')) = x''$	transitivity of = g, h
j. $x'' = x$	symmetry and transitivity of = h, d

step	formula	justification	plan
1	$(\forall x)(\forall y)(\forall z)\ f(x, f(y, z)) = f(f(x, y), z)$	G1	
2	$(\forall x)\ f(x, u) = x$	G2R	
3	$(\forall x)\ f(x, i(x)) = u$	G3R	
4	x		
5	$f(i(x), i(i(x))) = u$	∀e 3 [i(x)/x]	a.
6	$f(x, u) = f(x, u)$	=i	
7	$f(x, f(i(x), i(i(x)))) = f(x, u)$	=e sub lhs 5 for u in lhs 6	b.
8	$(\forall x)\ f(u, x) = x$	G2L	
9	$f(u, x) = x$	∀e 7	c.
10	$f(x, f(i(x), i(i(x)))) = x$	transitivity of = 7, 9	d.
11	$(\forall y)(\forall z)\ f(x, f(y, z)) = f(f(x, y), z)$	∀e 1 [x/x]	
12	$(\forall z)\ f(x, f(i(x), z)) = f(f(x, i(x)), z)$	∀e 11 [i(x)/y]	
13	$f(x, f(i(x), i(i(x)))) = f(f(x, i(x)), i(i(x)))$	∀e 12 [i(i(x))/z]	e.
14	$f(x, i(x)) = u$	∀e 3 [x/x]	f.
15	$f(x, f(i(x), i(i(x)))) = f(u, i(i(x)))$	subst. for lhs 14 in rhs 13	g.
16	$f(u, i(i(x))) = i(i(x))$	∀e 8 [i(i(x))/x]	h.
17	$f(x, f(i(x), i(i(x)))) = i(i(x))$	trans. of = 15, 16	i.
18	$x = i(i(x))$	trans. symm. of = 10, 17	j.
19	$(\forall x)\ x = i(i(x))$	∀i 4-18	

English language version of 1: “Let x be an arbitrary element of the group. Then the inverse $i(x)$ is an element, and it has an inverse $i(i(x))$, meaning that $f(i(x), i(i(x))) = u$. Apply f to both sides with x as the first argument, and the sides of this equation as the second to get $f(x, f(i(x), i(i(x)))) = f(x, u)$. By the associative property on the left-hand side and the definition of unit on the right, we have $f(f(x, i(x)), i(i(x))) = x$. But $f(x, i(x)) = u$, so substituting on the left, obtain $f(u, i(i(x))) = x$. By theorem G2L, we have $i(i(x))$ on the left, so $i(i(x)) = x$. But x was an arbitrary element, so $(\forall x)\ (i(i(x)) = x)$.”

2. Give a natural deduction proof of the following, based on the Number Theory axioms given in class, using the *rule* form of induction, rather than the axiom form:

$$(\forall x)\ (\forall y)\ (\forall z)\ (x+(y+z)) = ((x+y) + z)$$

step	formula	justification
1	$(\forall x)\ (0 + x) = x$	Theorem 1
2	y	
3	z	
4	$(0 + (y + z)) = ((0 + y) + z)$	∀e 1 [(y + z) / x]
5	$(\forall z)\ (0 + (y + z)) = ((0 + y) + z)$	∀i 3-4
6	$(\forall y)\ (\forall z)\ (0 + (y + z)) = ((0 + y) + z)$	∀i 2-5
7	$(\forall y)\ (\forall z)\ (x + (y + z)) = ((x + y) + z)$	inductive assumption
8	y	
9	z	
10	$(\forall z)\ (x + (y + z)) = ((x + y) + z)$	∀e 7 [y / y]
11	$(x + (y + z)) = ((x + y) + z)$	∀e 10 [z / z]
12	$S(x + (y + z)) = S((x + y) + z)$	=subst 11 as args of S
13	$(\forall y)\ (\forall x)\ (S(x) + y) = S(x + y)$	Theorem 2
14	$(\forall x)\ (S(x) + z) = S(x + z)$	∀e 13 [z / y]
15	$S(x + y) + z = S((x +y) + z)$	∀e 14 [(x + y) / x]
16	$S(x) + y = S(x + y)$	∀e 13 [y / y][x / x]
17	$S(x + y) = S(x) + y$	symm. of = 16
18	$S(x + y) + z = (S(x) + y) + z$	= subst. 17 as args of +
19	$S(x + (y + z)) = (S(x) + y) + z$	trans. of = 12, 15, 18
20	$S(x) + (y + z) = S(x + (y + z))$	∀e 13 [(y + z) / y]
21	$S(x) + (y + z) = (S(x) + y) + z$	trans. of = 20, 19
22	$(\forall z)\ (S(x) + (y + z)) = ((S(x) + y) + z)$	∀i 9-21
23	$(\forall y)\ (\forall z)\ (S(x) + (y + z)) = ((S(x) + y) + z)$	∀i 8-22
24	$(\forall x)\ (\forall y)\ (\forall z)\ (S(x) + (y + z)) = ((S(x) + y) + z)$	Induction rule 6, 7-23

3. **Hein page 415 Exercise 10.b.**

   Find all possible interpretations of W: $\exists x\, p(x) \rightarrow \forall x\, p(x)$ over the domain D = {a, b}. Give the truth value of W over each interpretation.

   There are four interpretations $\mu(p)$ of p, the only predicate symbol, therefore four interpretations (D, μ) total, where $\mu(p) = \{\}$, $\mu(p) = \{(a)\}$, $\mu(p) = \{(b)\}$, and $\mu(p) = \{(a), (b)\}$, respectively. Since the main sub-formulas of W, namely $\exists x\, p(x)$ and $\forall x\, p(x)$, are both closed (have no free variables), there is only one assignment to be considered for them: the empty assignment, call it α_0. For the sub-sub-formula p(x), the assignments consistent with α_0 on {} are α_1 where $\alpha_1(x) = a$, and α_2 where $\alpha_2(x) = b$.

   i. Interpretation $\mu(p) = \{\}$: $\alpha_0(\exists x\, p(x)) = F$, since for neither assignment α_1 or α_2 is $\alpha_i(p(x)) = T$.

   Therefore, from the definition of →, W is **T** for this interpretation.

   ii. Interpretation $\mu(p) = \{(a)\}$: $\alpha_0(\exists x\, p(x)) = T$, since $\alpha_1 (p(x)) = T$. However, $\alpha(\forall x\, p(x)) = F$, since $\alpha_2 (p(x)) = F$.

   Therefore, from the definition of →, W is **F** for this interpretation.

   iii. Interpretation $\mu(p) = \{(b)\}$: Same analysis as in ii with α_1 and α_2 interchanged.

   iv. Interpretation $\mu(p) = \{(a), (b)\}$: $\alpha_0(\exists x\, p(x)) = T$ as in ii, and since $\alpha_1(p(x)) = T$ and $\alpha_2(p(x)) = T$, $\alpha_0(\forall x\, p(x)) = T$.

   Therefore, from the definition of →, W is **T** for this interpretation.

4. **Hein page 415 Exercise 11.d**, with the added restriction that the l.h.s. of the implication must be true in the model.
   Find a model for this formula: $\forall x\, \exists y\, p(x, y) \rightarrow \exists y \forall x\, p(x, y)$.
   Such a model is (Δ, μ) where $\Delta = \{0\}$, $\mu(p) = \{(0, 0)\}$. Since every assignment for p assigns 0 to both x and y, both sides of the implication are **T**, hence the implication is **T**.
   Another way to look at it is through the enumeration of the possible interpretations, of which there are two:

p(0, 0)	$\exists y\, p(x, y)$	$\forall x\, p(x, y)$	$\forall x\, \exists y\, p(x, y)$	$\exists y \forall x\, p(x, y)$
F	F	F	F	F
T	T	T	T	T

The second interpretation is the model.

5. **Hein page 415 Exercise 12.e**, with the added restriction that the l.h.s. of the implication must be true in the model.
Find a countermodel for this formula: $\forall x \exists y\, p(x, y) \to \exists y \forall x\, p(x, y)$.
Such an interpretation is (Δ, μ) where $\Delta = \{0, 1\}$, $\mu(p) = \{(0, 0), (1, 1)\}$. The l.h.s. of the implication is **T,** since for any assignment for x, we can always choose the value of y to be the same as the value of x in order to make p(x, y) true. However the r.h.s. of the implication is F, since there is at least one assignment that makes it false, namely the one where y has the value opposite that of x.
Another way to look at it is through the enumeration of the possible interpretations, of which there are 16, along with the values of the sub-formulas under the two possible assignments for x and y:

p(0,0)	p(0,1)	p(1,0)	p(1,1)	$\exists y\, p(x, y)$ where $x \to 0$	$\exists y\, p(x, y)$ where $x \to 1$	$\forall x\, p(x, y)$ where $y \to 0$	$\forall x\, p(x, y)$ where $y \to 1$
F	F	F	F	F	F	F	F
F	F	F	T	F	T	F	F
F	F	T	F	F	T	F	F
F	F	T	T	F	T	F	F
F	T	F	F	T	F	F	F
F	T	F	T	T	T	F	T
F	T	T	F	T	T	F	F
F	T	T	T	T	T	F	T
T	F	F	F	T	F	F	F
T	**F**	**F**	**T**	**T**	**T**	**F**	**F**
T	F	T	F	T	T	T	F
T	F	T	T	T	T	T	F
T	T	F	F	T	F	F	F
T	T	F	T	T	T	F	T
T	T	T	F	T	T	T	F
T	T	T	T	T	T	T	T

Note that $\forall x \exists y\, p(x, y)$ is T only when both columns for $\exists y\, p(x, y)$ are T, whereas $\exists y \forall x\, p(x, y)$ is F only when both columns for $\forall x\, p(x, y)$ are F. So there is only one interpretation out of 16 that works, for this 2-element domain.

6. **Hein page 415 Exercise 14.b**.

Show that W: $\forall x\, p(x, x) \to \forall x \forall y \forall z\, (p(x, y) \vee p(x, z) \vee p(y, z))$ is true in any interpretation having a domain of exactly two elements.

Assume that we have an interpretation in which the l.h.s. of $\to$ is T. Then the r.h.s. atomic formulas p(x, y), p(x, z), or p(y, z) can all be false only if each of the variables x, y, and z can be assigned distinct values. If any two of the variables x, y, z have the same value in an assignment, then at least one of the atomic formulas is T.

For the special case of a 2-element domain, there is no way to assign three distinct values to three variables (a special case of the "pigeonhole principle"). Hence the r.h.s. cannot be made F and the formula must be T.

7. **Hein page 415 Exercise 14.c**.

To show that W in the previous problem is not valid, it suffices to find an interpretation inducing the value F. Consider the interpretation with domain $\{0, 1, 2\}$, where $p = \{(0, 0), (1, 1), (2, 2)\}$. The l.h.s. of the implication will be T, but the r.h.s. won't, since the assignment $x \to 0$, $y \to 1$, $z \to 2$ is such that $p(x, y) \lor p(x, z) \lor p(y, z)$ has the induced value F.

8. **Hein page 416 Exercise 16.g.**

Prove that this formula is valid: $\forall x(A(x) \to B(x)) \to (\forall x A(x) \to \exists x\, B(x))$

Proof: Consider an arbitrary interpretation I with assignment α for which $\alpha(\forall x(A(x) \to B(x))) = T$. Suppose further that $\alpha(\forall x A(x)) = T$. Since the domain is non-empty, we can let d be a particular element of it. In particular $\alpha(A(d)) = T$ by our supposition, and since $\alpha(\forall x(A(x) \to B(x))) = T$, also $\alpha(B(d)) = T$. Therefore $\alpha(\exists x\, B(d)) = T$, hence $\alpha(\forall x A(x) \to \exists x\, B(x))) = T$. Therefore the induced value for the entire formula is T.

9. **Hein page 416 Exercise 19.**

Prove that a formula of the form $A \to B$ is valid iff whenever A is valid B is valid.

Proof: A formula is valid iff it is true for every interpretation and assignment.

Suppose $A \to B$ is valid. Suppose further that A is valid, to show that B must be also. Let (I, α) be an arbitrary interpretation and assignment such that $\alpha(A) = T$. Since $\alpha(A \to B) = \alpha(B)$ when $\alpha(A) = T$, and $\alpha(A \to B) = T$ because $A \to B$ is valid, we must also have $\alpha(B) = T$. But since (I, α) is arbitrary, we have that B is valid.

Conversely, suppose that whenever A is valid, B must be valid. We want to show that $A \to B$ is valid. We must show that for any arbitrary interpretation and assignment (I, α), $\alpha(A \to B) = T$. Suppose that (i) there is an interpretation and assignment (I, α) such that $\alpha(A) = F$. Then $\alpha(A \to B) = T$ from the truth table for $\to$. On the other hand, if (ii) there is no interpretation and assignment where $\alpha(A) = F$, then certainly A is valid. Then by assumption B is valid, and so for any interpretation and assignment $\alpha(B) = T$. Thus $\alpha(A \to B) = T$ from the truth table for $\to$. So in both cases (i) and (ii), $\alpha(A \to B) = T$. Thus $A \to B$ must be valid.

10. **Bonus (Extra credit)**: Give a natural deduction proof of formula 7.6 on Hein page 420.

$$\exists x(p(x) \rightarrow q(x)) \equiv (\forall x\ p(x) \rightarrow \exists x\ q(x))$$

1.	$\exists x(p(x) \rightarrow q(x))$	Assumption
2.	$x_0 \quad p(x_0) \rightarrow q(x_0)$	Assumption
3.	$\forall x\ p(x)$	Assumption
4.	$p(x_0)$	∀e 3
5.	$q(x_0)$	→e 4, 2
6.	$\exists x\ q(x)$	∃i 5
7.	$\forall x\ p(x) \rightarrow \exists x\ q(x)$	→i 3-7
8.	$\forall x\ p(x) \rightarrow \exists x\ q(x)$	∃e 2-7
9.	$\forall x\ p(x) \rightarrow \exists x\ q(x)$	Assumption
10.	$\forall x\ p(x) \vee \neg \forall x\ p(x)$	LEM
11.	$\forall x\ p(x)$	Assumption
12.	$\exists x\ q(x)$	→e 11, 9
13.	$x_1\ q(x)$	Assumption
14.	$p(x_1)$	Assumption
15.	$q(x_1)$	Copy 13
16.	$p(x_1) \rightarrow q(x_1)$	→i 14-15
17.	$\exists x(p(x) \rightarrow q(x))$	∃i 16
18.	$\exists x(p(x) \rightarrow q(x))$	∃e 12 13-17
19.	$\neg \forall x\ p(x)$	Assumption
20.	$\exists x\ \neg p(x)$	DeMorgan 19
21.	$x_2\ \neg p(x_2)$	Assumption
22.	$p(x_2)$	Assumption
23.	$\bot$	¬e 22, 21
24.	$q(x_2)$	⊥e 23
25.	$p(x_2) \rightarrow q(x_2)$	→i 22-24
26.	$\exists x(p(x) \rightarrow q(x))$	∃i 25
27.	$\exists x(p(x) \rightarrow q(x))$	∃e 20 21-26
28.	$\exists x(p(x) \rightarrow q(x))$	∨e 10, 11-18, 19-28
29.	$\exists x(p(x) \rightarrow q(x)) \equiv (\forall x\ p(x) \rightarrow \exists x\ q(x))$	≡i 1-8, 9-28