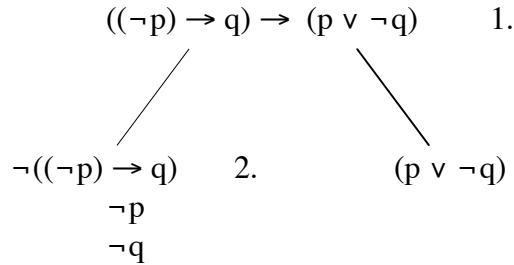


**CS 81 Assignment 5 for Wed., Feb. 23**

“Tree method” refers to the method described in the class notes with the corresponding title.

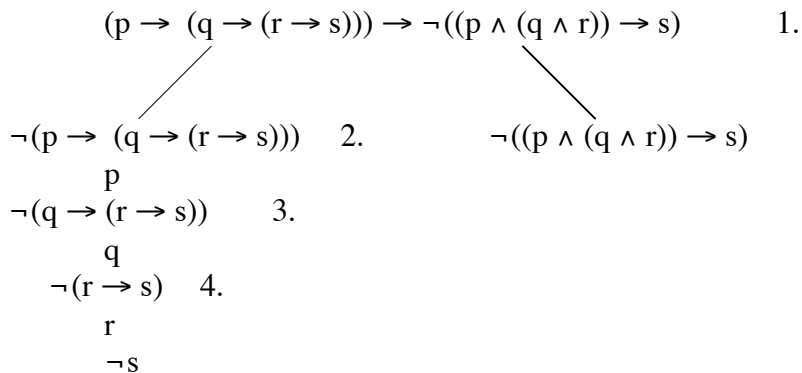
1. Determine whether or not these formulas are satisfiable, using the tree method:
1.  $((\neg p) \rightarrow q) \rightarrow (p \vee \neg q)$

I’m putting a number by the formulas to show the order in which they were checked.



The leftmost path is open, so the formula is **satisfiable** (by  $p = F, q = F$ ).

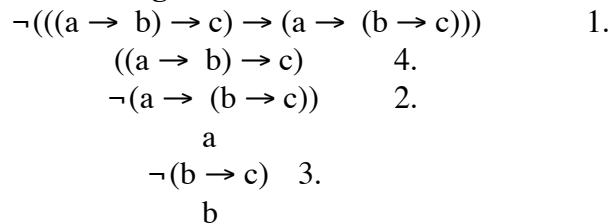
2.  $(p \rightarrow (q \rightarrow (r \rightarrow s))) \rightarrow \neg((p \wedge (q \wedge r)) \rightarrow s)$



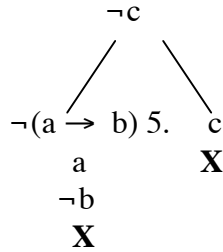
The leftmost path is open, so the formula is **satisfiable** (by  $p = q = r = T, s = F$ ).

2. Determine whether or not these formulas are tautologies, using the tree method:
1.  $((a \rightarrow b) \rightarrow c) \rightarrow (a \rightarrow (b \rightarrow c))$

The tree for the **negated** formula is:



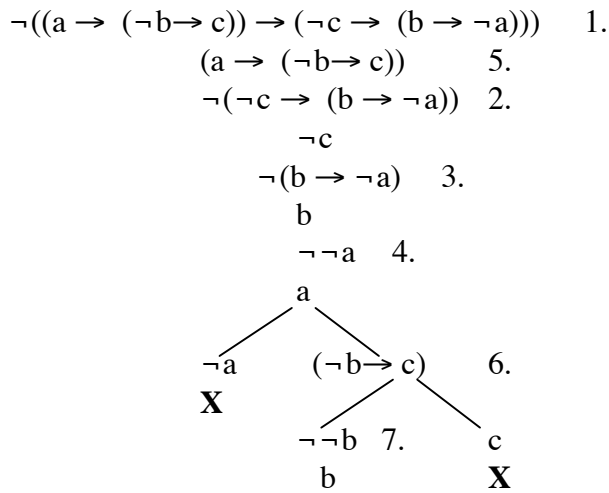
(continued next page)



All paths are closed, so the negation of the original formula is not satisfiable. Therefore the original formula is a **tautology**.

2.  $(a \rightarrow (\neg b \rightarrow c)) \rightarrow (\neg c \rightarrow (b \rightarrow \neg a))$

The negated formula is:



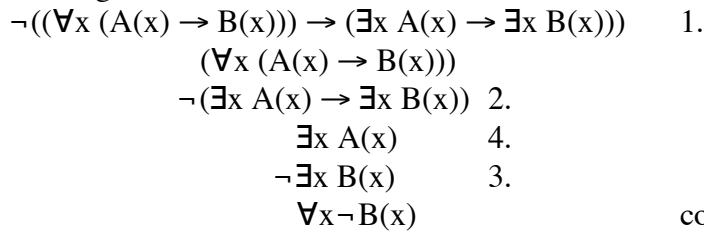
The middle path is open, therefore the negated formula is satisfiable. Thus the original formula is **not a tautology** ( $a = T, b = T, c = F$  induces the value F).

3. Show that these formulas are valid, using the tree method:

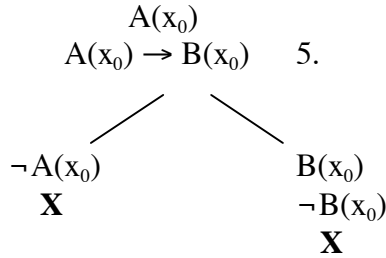
1. Hein p. 416, 16f

$$(\forall x (A(x) \rightarrow B(x))) \rightarrow (\exists x A(x) \rightarrow \exists x B(x))$$

The negated formula is:



continued next page



All paths are closed, so the negation of the original formula is not satisfiable. Therefore the original formula is valid.

2. Hein p. 416, 16g

$$(\forall x (A(x) \rightarrow B(x))) \rightarrow (\forall x A(x) \rightarrow \exists x B(x))$$

The negated formula is:

$$\neg((\forall x (A(x) \rightarrow B(x))) \rightarrow (\forall x A(x) \rightarrow \exists x B(x))) \quad 1.$$

$$(\forall x (A(x) \rightarrow B(x)))$$

$$\neg(\forall x A(x) \rightarrow \exists x B(x)) \quad 2.$$

$$\forall x A(x)$$

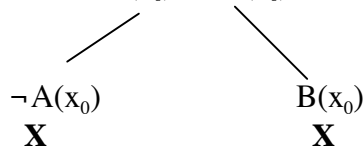
$$\neg \exists x B(x) \quad 3.$$

$$\forall x \neg B(x)$$

$$A(x_0)$$

$$\neg B(x_0)$$

$$A(x_0) \rightarrow B(x_0) \quad 4.$$



All paths are closed, so the negation of the original formula is not satisfiable. Therefore the original formula is valid.

4. Hein p. 416 17 c, using the tree method.

$$\exists x \forall y (p(x, y) \wedge \neg p(x, y)) \quad 1.$$

$$\forall y (p(x_0, y) \wedge \neg p(x_0, y))$$

$$(p(x_0, y_0) \wedge \neg p(x_0, y_0))$$

$$p(x_0, y_0)$$

$$\neg p(x_0, y_0)$$

$\mathbf{X}$

All paths are closed, therefore the formula is not satisfiable.

5. Hein p. 486 3b.

$$\{x < y\}$$

$$\mathbf{y := y + x;}$$

$$\mathbf{x := y - x;}$$

$$\mathbf{y := y - x;}$$

$$\{y < x\}$$

Work from the bottom up using the assignment rule, to get the fully annotated program:

```

    {x < y}
    {((y+x)-((y+x)-x)) < ((y+x)-x)}
    y := y + x;
    {(y-(y-x)) < (y-x)}
    x := y - x;
    {(y-x) < x}
    y := y - x;
    {y < x}

```

The second line follows from the first by algebraic simplification of the second.

An alternate, line-by-line derivation:

1.  $\{(y-x) < x\} \mathbf{y := y - x}; \{y < x\}$  Assignment
2.  $\{(y-(y-x)) < (y-x)\} \mathbf{x := y - x}; \{(y-x) < x\}$  Assignment
3.  $\{(y-(y-x)) < (y-x)\} \mathbf{x := y - x}; \mathbf{y := y - x}; \{y < x\}$  Composition 2, 1
4.  $\{((y+x)-((y+x)-x)) < ((y+x)-x)\} \mathbf{y := y + x}; \{(y-(y-x)) < (y-x)\}$  Assignment
5.  $\{x < y\} \mathbf{y := y + x}; \{(y-(y-x)) < (y-x)\}$  Assumption strengthening 4
6.  $\{x < y\} \mathbf{y := y + x}; \mathbf{x := y - x}; \mathbf{y := y - x}; \{y < x\}$  Composition 5, 3

6. Hein p. 486 5b. The domain is the set of integers.

```

    { true }
    if( x < y )
        y := y - 1;
    else
        x := -x;
        y := -y;
    { x ≤ y }

```

Examine the two branches separately, starting with the same expectation and working backwards. We then get

```

    { true }
    if( x < y )
        { x ≤ (y - 1) }
        y := y - 1;
        { x ≤ y }
    else
        { -x ≤ -y }
        x := -x;
        { x ≤ -y }
        y := -y;

```

$$\{ x \leq y \}$$

$$\{ x \leq y \}$$

In the domain of integers,  $x \leq (y - 1)$  is equivalent to  $x < y$ .

For numbers in general,  $-x \leq -y$  is equivalent to  $x \geq y$ , which is equivalent to  $\neg(x < y)$ . So using the assumption-strengthening rule, we have triples for the two branches:

$$\begin{array}{ll} \{ \text{true} \wedge x < y \} & \{ \text{true} \wedge \neg(x < y) \} \\ \mathbf{y := y - 1;} & \mathbf{x := -x;} \\ \{ x \leq y \} & \mathbf{y := -y;} \\ & \{ x \leq y \} \end{array}$$

Which enables us to infer, using the **if** rule:

$$\begin{array}{l} \{ \text{true} \} \\ \mathbf{if( x < y )} \\ \quad \mathbf{y := y - 1;} \\ \mathbf{else} \\ \quad \mathbf{x := -x;} \\ \quad \mathbf{y := -y;} \\ \{ x \leq y \} \end{array}$$

An alternate, line-by-line derivation:

1.  $\{ x \leq (y - 1) \} \mathbf{y := y - 1}; \{ x \leq y \}$  Assignment
2.  $\{ \text{true} \wedge x < y \} \mathbf{y := y - 1}; \{ x \leq y \}$  Assumption strengthening 1
3.  $\{ x \leq -y \} \mathbf{y := -y}; \{ x \leq y \}$  Assignment
4.  $\{ -x \leq -y \} \mathbf{x := -x}; \{ x \leq -y \}$  Assignment
5.  $\{ -x \leq -y \} \mathbf{x := -x}; \mathbf{y := -y}; \{ x \leq y \}$  Composition 4, 3
6.  $\{ \text{true} \wedge \neg(x < y) \} \mathbf{x := -x}; \mathbf{y := -y}; \{ x \leq y \}$  Assumption strengthening 5
7.  $\{ \text{true} \} \mathbf{if( x < y ) y := y - 1; else x := -x; y := -y; fi}; \{ x \leq y \}$  If 2, 5

7. Hein p. 487 7

$$\begin{array}{l} \{ x \geq y \wedge \text{even}(x-y) \} \\ \mathbf{while( x \neq y ) do} \\ \quad \mathbf{x := x - 1;} \\ \quad \mathbf{y := y + 1;} \\ \mathbf{od} \\ \{ x \geq y \wedge \text{even}(x-y) \wedge x = y \} \end{array}$$

The proposed invariant is:  $x \geq y \wedge \text{even}(x-y)$ . In order to use the while rule to get the above triple, it suffices to derive the triple

$$\{ x \geq y \wedge \text{even}(x-y) \wedge x \neq y \} \mathbf{x := x - 1; y := y + 1; \{ x \geq y \wedge \text{even}(x-y) \}}$$

1.  $\{x \geq (y+1) \wedge \text{even}(x-(y+1))\} \mathbf{y := y + 1}; \{x \geq y \wedge \text{even}(x-y)\}$  Assignment
2.  $\{((x-1) \geq (y+1) \wedge \text{even}((x-1)-(y+1)))\} \mathbf{x := x - 1}; \{x \geq (y+1) \wedge \text{even}(x-(y+1))\}$  “
3.  $\{x \geq (y+2) \wedge \text{even}(x-y-2)\} \mathbf{x := x - 1}; \{x \geq (y+1) \wedge \text{even}(x-(y+1))\}$  Strengthen 2
4.  $\{x \geq y \wedge \text{even}(x-y) \wedge x \neq y\} \mathbf{x := x - 1}; \{x \geq (y+1) \wedge \text{even}(x-(y+1))\}$  Strengthen 3
5.  $\{x \geq (y+2) \wedge \text{even}(x-y-2)\} \mathbf{x := x - 1; y := y + 1}; \{x \geq y \wedge \text{even}(x-y)\}$  Comp. 3 4

**Important:** To get step 4, we need to argue that

$$(x \geq y \wedge \text{even}(x-y) \wedge x \neq y) \rightarrow ((x \geq (y+2) \wedge \text{even}(x-y-2)))$$

For any number,  $\text{even}(z) \rightarrow \text{even}(z-2)$ , by definition of even, which gives us  $\text{even}(x-y) \rightarrow \text{even}(x-y-2)$ .

To get  $(x \geq y \wedge x \neq y) \rightarrow (x \geq (y+2))$ , we note that

$$(x \geq y \wedge x \neq y) \rightarrow (x > y) \text{ for numbers in general.}$$

Because  $\text{even}(x-y)$ ,  $x$  must exceed  $y$  by at least 2. Exceeding by only 1 is not possible. Therefore

$$(x \geq y \wedge x \neq y) \rightarrow (x \geq (y+2))$$