Equality in the Predicate Calculus

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Equality as a Predicate

- In predicate logic, equality is often given special dispensation.
- It is represented in infix form ($x = y$, instead of $e(x, y)$).
- When we get to semantics, it is always interpreted to mean equality (which is to say, identity).

ND Inference Rules for Equality

= Introduction Rule:

$$\frac{}{t = t} \quad =\text{i}$$

(t is any term)

The premise set is empty (just as for the L.E.M., for example).

ND Inference Rules for Equality

- Equality Elimination

$$\frac{t_1 = t_2 \qquad \varphi[t_1 / x]}{\varphi[t_2 / x]} \; =e$$

where t_1 and t_2 are both free to replace x in φ

Example: Derivation using Equality Rules

- Derive $\vdash (\forall x)(x = x)$

1. x_0
2. $x_0 = x_0$ =i
3. $(\forall x)(x = x)$ ∀i 1-2

Example: Derivation using Equality Rules

- Derive $\vdash (\forall x)(\forall y)\,((x = y) \rightarrow (y = x))$

1.	x_0	
2.	y_0	
3.	$x_0 = y_0$	Assumption
4.	$x_0 = x_0$	=i
5.	$y_0 = x_0$	=e 3, 4 (with φ: $x = x_0$)
6.	$(x_0 = y_0) \rightarrow (y_0 = x_0)$	$\rightarrow$i 3-5
7.	$(\forall y)\,((x_0 = y) \rightarrow (y = x_0))$	$\forall$y i 2-6
8.	$(\forall x)(\forall y)\,((x = y) \rightarrow (y = x))$	$\forall$x i 1-7

Example: Derivation using Equality Rules

- Derive $\vdash (\forall x)(\forall y)(\forall z)(((x = y)\wedge(y = z)) \rightarrow (x = z))$

1.	x_0	
2.	y_0	
3.	z_0	
4.	$(x_0 = y_0)\wedge(y_0 = z_0)$	Assumption
5.	$(x_0 = y_0)$	$\wedge e_1$ 4
6.	$(y_0 = z_0)$	$\wedge e_2$ 4
7.	$(y_0 = x_0)$	previous deriv. 6
8.	$(x_0 = z_0)$	$=e$ 7, 6 (φ: $x = z_0$)
9.	$((x_0 = y_0)\wedge(y_0 = z_0)) \rightarrow (x_0 = z_0)$	$\rightarrow i$ 4-8
10.	$(\forall z)\ (((x_0 = y_0)\wedge(y_0 = z)) \rightarrow (x_0 = z))$	$\forall z$ i 3-9
11.	$(\forall y)(\forall z)\ (((x_0 = y)\wedge(y = z)) \rightarrow (x_0 = z))$	$\forall y$ i 2-10
12.	$(\forall x)(\forall y)(\forall z)\ (((x = y)\wedge(y = z)) \rightarrow (x = z))$	$\forall x$ i 1-11

Equality "Axioms" (These can all be derived using the equality rules.)

- $(\forall x)\ (x = x)$ — Reflexive property of =
- $(\forall x)(\forall y)\ ((x = y) \rightarrow (y = x))$ — Symmetric property of =
- $(\forall x)(\forall y)(\forall z)(((x = y) \wedge (y = z)) \rightarrow (x = z))$ — Transitive property of =
- $(\forall x_1) \ldots (\forall x_n)(\forall y_1) \ldots (\forall y_n)$ — Equals for equals (func.)

  $(((x_1 = y_1) \wedge \ldots \wedge (x_n = y_n)) \rightarrow (\mathbf{f}(x_1,\ldots,x_n) = \mathbf{f}(y_1,\ldots,y_n)))$

  for each n-ary function symbol **f**.
- $(\forall x_1) \ldots (\forall x_n)(\forall y_1) \ldots (\forall y_n)$ — Equals for equals (pred.)

  $(((x_1 = y_1) \wedge \ldots \wedge (x_n = y_n)) \rightarrow (\mathbf{p}(x_1,\ldots,x_n) \equiv \mathbf{p}(y_1,\ldots,y_n)))$

  for each n-ary predicate symbol **p**.