Example of the construction in the completeness proof, using $\neg(p \rightarrow q)$

- We are trying to demonstrate that, for an arbitrary formula η and assignment α that:
  if $\alpha(\eta)$ = T, then there is a proof of … $|-\ \eta$, whereas
  if $\alpha(\eta)$ = F, then there is a proof of … $|-\ \neg\eta$.
  Here … represents a set of proposition letters or their negations such that α of each is true.
- Example: $\neg(p \rightarrow q)$

$\alpha(p)$	$\alpha(q)$	$\alpha(\neg(p \rightarrow q))$	Premises	Derive
F	F	F	$\neg p$, $\neg q$	$\neg(\neg(p \rightarrow q))$
F	T	F	$\neg p$, q	$\neg(\neg(p \rightarrow q))$
T	F	T	p, $\neg q$	$\neg(p \rightarrow q)$
T	T	F	p, q	$\neg(\neg(p \rightarrow q))$

$\alpha(p)$	$\alpha(q)$	$\alpha(\neg(p \rightarrow q))$	Premises	Derive
F	F	F	$\neg p, \neg q$	$\neg(\neg(p \rightarrow q))$

- $\alpha(p \rightarrow q)$ = T, so by the induction hypothesis, there is a proof
  $\neg p, \neg q \vdash p \rightarrow q$
- From the meaning of $\rightarrow$, such a proof can be constructed from a proof of either
  $\neg p, \neg q \vdash \neg p$
  or $\neg p, \neg q \vdash q$
- We have the first (but not the second). Thus the mini-proof is

1.	$\neg p$	Premise
2.	$\neg q$	Premise
3.	p	Assumption
4.	$\bot$	$\neg$e 3, 1
5.	q	$\bot$e 4
6.	$p \rightarrow q$	$\rightarrow$i 3-5
7.	$\neg(\neg(p \rightarrow q))$	$\neg\neg$i 6

$\alpha(p)$	$\alpha(q)$	$\alpha(\neg(p \rightarrow q))$	Premises	Derive
F	T	F	$\neg p,\ q$	$\neg(\neg(p \rightarrow q))$

- $\alpha(p \rightarrow q)$ = T, so by the induction hypothesis, there is a proof
  $\neg p, q \vdash p \rightarrow q$
- From the meaning of $\rightarrow$, such a proof can be constructed from a proof of either
  $\neg p, q \vdash \neg p$
  or $\neg p, q \vdash q$
- We have both, so just use the second. Thus the mini-proof is

1. $\neg p$ Premise
2. q Premise
3. p Assumption
4. q copy 2
5. $p \rightarrow q$ $\rightarrow$i 3-4
6. $\neg(\neg(p \rightarrow q))$ $\neg\neg$i 5

$\alpha(p)$	$\alpha(q)$	$\alpha(\neg(p \rightarrow q))$	Premises	Derive
T	F	T	$p, \neg q$	$\neg(p \rightarrow q)$

- $\alpha(p \rightarrow q)$ = F, so by the induction hypothesis, there is a proof
  $$p, \neg q \vdash \neg(p \rightarrow q)$$
- From the meaning of $\rightarrow$, making $p \rightarrow q$ false requires **both**
  $$p, \neg q \vdash p$$
  and $$p, \neg q \vdash \neg q$$
- Fortunately, these are both trivial Thus the mini-proof is

1.	p	Premise
2.	$\neg q$	Premise
3.	$p \rightarrow q$	Assumption
4.	q	$\rightarrow$e 1, 3
5.	$\bot$	$\neg$e 4, 2
6.	$\neg(p \rightarrow q)$	$\neg$i 3-5

$\alpha(p)$	$\alpha(q)$	$\alpha(\neg(p \to q))$	Premises	Derive
T	T	F	p, q	$\neg(\neg(p \to q))$

- $\alpha(p \to q)$ = T, so by the induction hypothesis, there is a proof
  $p, q \vdash (p \to q)$
- From the meaning of $\to$, making $p \to q$ true requires **either**
  $p, q \vdash \neg p$
  or $p, q \vdash q$
- Only the second is possible. Thus the mini-proof is

1. p — Premise
2. q — Premise
3. p — Assumption
4. q — copy 2
5. $p \to q$ — $\to$i 3-4
6. $\neg(\neg(p \to q))$ — $\neg\neg$i 5