Mid-Term Solutions

Part 1

1. [5 points]

In propositional logic, explain the relationships between tautologies, satisfiable formulas, and unsatisfiable formulas.

Answer: Relative to the set of assignments of truth values to all proposition variables:
A tautology is a formula that has the induced value true for every assignment.
A formula is satisfiable if it has the induced value true for some assignment.
A formula is unsatisfiable if it has the induced value true for no assignment.

Therefore, a formula is a tautology iff its negation is unsatisfiable.

(The negation of a satisfiable formula that is not a tautology is also satisfiable, but not a tautology.)

2. [5 points]

a. Determine whether or not this formula is a tautology:

$$
\begin{aligned}
&((p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)) \\
\wedge\ &(q \vee \neg q) \\
\wedge\ &((p \rightarrow q) \vee (q \rightarrow r))
\end{aligned}
$$

Answer:
A conjunction is a tautology iff each of its conjuncts is a tautology.
Above, all conjuncts are tautologies, so the overall formula is a tautology.
The first conjunct is the familiar rule that an impliciation implies its contrapositive.
The second conjunct is the law of the excluded middle.
The third conjunct is a tautology by Quine's method: If q is T then $(p \rightarrow q)$ is T, while if q is F, then $(q \rightarrow r)$ is T.

b. Determine whether or not this formula is satisfiable:

$$
\begin{aligned}
&\neg(p \rightarrow ((p \rightarrow q) \rightarrow q)) \\
\vee\ &(p \rightarrow \neg p) \\
\vee\ &\neg((p \wedge (p \rightarrow q)) \rightarrow \neg(\neg q \wedge \neg p))
\end{aligned}
$$

Answer:
A disjunction is satisfiable iff one if its disjuncts is satisfiable.
Above the middle conjunct is satisfied when p is F, so the overall formula is satisfiable.

3. [5 points]

In predicate logic, please contrast the meanings of the following:

atomic formula

An atomic formula consists of a predicate applied to arguments that are terms (where terms are as defined below).

formula

This definition is recursive: A formula is either an atomic formula, a set of formulas connected by one of the propositional connectives, or a quantified variable followed by a formula.

predicate symbol

A predicate symbol is a symbol that stands for a predicate. A predicate gives the value true or false for each combination of its arguments.

term

This definition is recursive. A term is either a constant symbol, a variable symbol, or a function symbol applied to the appropriate number of terms as its arguments.

4. [5 points]

State in your own words the distinction between $\vdash$ and $\models$

a. [2/5 points] for propositional logic

   $\varphi_1, \ldots, \varphi_n \vdash \psi$ means that formula ψ is derivable from formulas $\varphi_1, \ldots, \varphi_n$ using the rules of natural deduction.

   $\varphi_1, \ldots, \varphi_n \models \psi$ means that every assignment that induces true in each of $\varphi_1, \ldots, \varphi_n$ also induces true in ψ.

b. [3/5 points] for predicate logic

   $\varphi_1, \ldots, \varphi_n \vdash \psi$ means that formula ψ is derivable from formulas $\varphi_1, \ldots, \varphi_n$ using the rules of natural deduction.

   $\varphi_1, \ldots, \varphi_n \models \psi$ means that every interpretation and assignment that induces true in each of $\varphi_1, \ldots, \varphi_n$ also induces true in ψ.

4. [5 points]

Define the terms *soundness* and *completeness* of a logical proof system.

A proof system is **sound** if only true formulas are derivable. More generally, it is sound provided:

$$\text{If } \varphi_1, \ldots, \varphi_n \vdash \psi \text{ then } \varphi_1, \ldots, \varphi_n \models \psi$$

A proof system is **complete** if every true formula is derivable. More generally, it is complete provided:

$$\text{If } \varphi_1, \ldots, \varphi_n \models \psi \text{ then } \varphi_1, \ldots, \varphi_n \vdash \psi$$

Part 2

1. [20 points]

Before each formula, state T or F indicating whether or not the formula is universally valid. It is not necessary to give proofs. An extra point is subtracted for wrong answers, so if you don't know, it is probably better not to guess.

a. __T___ $(\exists x\, \exists y\, R(x, y)) \rightarrow \exists y\, \exists x\, R(x, y)$

b. __F___ $(\forall x\, \exists y\, R(x, y)) \rightarrow \exists y \forall x\, R(x, y)$

c. __T___ $(\exists x\, \forall y\, R(x, y)) \rightarrow \forall y\, \exists x\, R(x, y)$

d. __F__ $(\forall x\, \exists y\, R(x, y)) \rightarrow \forall y\, \exists x\, R(x, y)$

e. __T__ $(\forall x\, \forall y\, (R(x, y) \wedge R(y, x))) \rightarrow \forall x\, R(x, x)$

f. __T__ $(\forall x\, \forall y\, R(x, y)) \rightarrow \exists x\, \exists y\, R(x, y)$

g. __F__ $((\exists x\, S(x)) \wedge (\exists x\, T(x))) \rightarrow \exists x\, (S(x) \wedge T(x))$

h. __F__ $(\forall x\, (S(x) \rightarrow T(x))) \rightarrow ((\neg \exists x\, S(x)) \vee \forall x\, T(x)))$

i. __T__ $((\forall x\, S(x)) \rightarrow (\forall x\, T(x))) \rightarrow ((\exists x\, \neg S(x)) \vee (\forall x\, T(x)))$

j. __T__ $(\forall x\, (S(x) \rightarrow T(x))) \rightarrow ((\exists x\, S(x)) \rightarrow (\exists x\, T(x)))$

2. [25 points]

Carefully provide a complete formal natural deduction proof of

$\forall x\ R(x, x)$

from the three premises listed below:

Provide justification for each step and do not skip steps.

1.	$\forall x\ \exists y\ R(x, y)$	Premise
2.	$\forall x\ \forall y\ \forall z\ ((R(x, y) \land R(y, z)) \rightarrow R(x, z))$	Premise
3.	$\forall x\ \forall y\ (R(x, y) \rightarrow R(y, x))$	Premise

Here is a proof formatted using the JAPE proof editor:

1:	$\forall x.\exists y.R(x,y)$	premise
2:	$\forall x.\forall y.\forall z.((R(x,y)\land R(y,z))\rightarrow R(x,z))$	premise
3:	$\forall x.\forall y.(R(x,y)\rightarrow R(y,x))$	premise
4:	actual i	assumption
5:	$\forall y.(R(i,y)\rightarrow R(y,i))$	$\forall$ elim 3,4
6:	$\forall y.\forall z.((R(i,y)\land R(y,z))\rightarrow R(i,z))$	$\forall$ elim 2,4
7:	$\exists y.R(i,y)$	$\forall$ elim 1,4
8:	actual i1	assumption
9:	$R(i,i1)$	assumption
10:	$R(i,i1)\rightarrow R(i1,i)$	$\forall$ elim 5,8
11:	$R(i1,i)$	$\rightarrow$ elim 10,9
12:	$R(i,i1)\land R(i1,i)$	$\land$ intro 9,11
13:	$\forall z.((R(i,i1)\land R(i1,z))\rightarrow R(i,z))$	$\forall$ elim 6,8
14:	$(R(i,i1)\land R(i1,i))\rightarrow R(i,i)$	$\forall$ elim 13,4
15:	$R(i,i)$	$\rightarrow$ elim 14,12
16:	$R(i,i)$	$\exists$ elim 7,8-15
17:	$\forall x.R(x,x)$	$\forall$ intro 4-16

3. [5 points]

The following meta-result was used in proving the completeness theorem for propositional logic:

$$\text{If} \vdash (\varphi_1 \rightarrow (\varphi_2 \rightarrow \ldots (\varphi_n \rightarrow \psi)) \ldots) \text{ then } \varphi_1, \ldots, \varphi_n \vdash \psi$$

Prove this by mathematical induction on n. Indicate clearly any derivation rules you use from the object (as opposed to meta) level system.

Note that for n = 0, the statement is "If $\vdash \psi$ then $\vdash \psi$."

Basis: n = 0

The statement is obviously correct.

Induction step:

Assume

$$\text{If} \vdash (\varphi_1 \rightarrow (\varphi_2 \rightarrow \ldots (\varphi_n \rightarrow \psi)) \ldots) \text{ then } \varphi_1, \ldots, \varphi_n \vdash \psi$$

Suppose for the n+1 formulas $\varphi_1, \ldots, \varphi_n$ we have

$$\vdash (\varphi_1 \rightarrow (\varphi_2 \rightarrow \ldots (\varphi_n \rightarrow (\varphi_{n+1} \rightarrow \psi))) \ldots)$$

to show:

$$\varphi_1, \ldots, \varphi_{n+1} \vdash \psi$$

But $(\varphi_{n+1} \rightarrow \psi)$ is just another formula that can replace ψ in the inductive assumption, so by that inductive assumption, we have

$$\varphi_1, \ldots, \varphi_n \vdash (\varphi_{n+1} \rightarrow \psi)$$

If we include φ_{n+1} in the premises:

$$\varphi_1, \ldots, \varphi_n, \varphi_{n+1} \vdash (\varphi_{n+1} \rightarrow \psi)$$

But then by **the → elimination rule**, we also have

$$\varphi_1, \ldots, \varphi_n, \varphi_{n+1} \vdash \psi$$

exactly as desired.

4. [30 points]

$\{ n \geq 0 \}$
$\{(0 = 0^3) \wedge (1 = (0+1)^3 - 0^3) \wedge (6 = 6*0+6) \wedge (0 \leq n)\}$

i := 0;
$\{(0 = i^3) \wedge (1 = (i+1)^3 - i^3) \wedge (6 = 6i+6) \wedge (i \leq n)\}$

x := 0;
$\{(x = i^3) \wedge (1 = (i+1)^3 - i^3) \wedge (6 = 6i+6) \wedge (i \leq n)\}$

y := 1;
$\{(x = i^3) \wedge (y = (i+1)^3 - i^3) \wedge (6 = 6i+6) \wedge (i \leq n)\}$

z := 6;
$\{(x = i^3) \wedge (y = (i+1)^3 - i^3) \wedge (z = 6i+6) \wedge (i \leq n)\}$

while(i < n)

$\{(x = i^3) \wedge (y = (i+1)^3 - i^3) \wedge (z = 6i+6) \wedge (i \leq n) \wedge (i < n) \}$
$\{(x = i^3) \wedge (y = (i+1)^3 - i^3) \wedge (z = 6i+6) \wedge ((i+1) \leq n) \}$
$\{(x = (i+1)^3 - ((i+1)^3 - i^3)) \wedge (y = (i+1)^3 - i^3) \wedge (z = 6i+6) \wedge ((i+1) \leq n) \}$
$\{(x = (i+1)^3 - y) \wedge (y = (i+1)^3 - i^3) \wedge (z = 6i+6) \wedge ((i+1) \leq n) \}$
$\{(x+y = (i+1)^3) \wedge (y = (i+1)^3 - i^3) \wedge (z = 6i+6) \wedge ((i+1) \leq n) \}$

x := x + y;
$\{(x = (i+1)^3) \wedge (y = (i+1)^3 - i^3) \wedge (z = 6i+6) \wedge ((i+1) \leq n) \}$
$\{(x = (i+1)^3) \wedge (y = (i^3 + 3i^2 + 3i + 1) - i^3) \wedge (z = 6i+6) \wedge ((i+1) \leq n) \}$
$\{(x = (i+1)^3) \wedge (y = (3i^2 + 3i + 1)) \wedge (z = 6i+6) \wedge ((i+1) \leq n) \}$
$\{(x = (i+1)^3) \wedge (y = (3i^2 + 9i + 7) - (6i+6)) \wedge (z = 6i+6) \wedge ((i+1) \leq n) \}$
$\{(x = (i+1)^3) \wedge (y = (i^3 + 6i^2 + 12i + 8) - (i^3 + 3i^2 + 3i + 1) - (6i+6))$
$\wedge (z = 6i+6) \wedge ((i+1) \leq n) \}$
$\{(x = (i+1)^3) \wedge (y = (i+2)^3 - (i+1)^3 - (6i+6)) \wedge (z = 6i+6) \wedge ((i+1) \leq n) \}$
$\{(x = (i+1)^3) \wedge ((y+(6i+6)) = (i+2)^3 - (i+1)^3) \wedge (z = 6i+6) \wedge ((i+1) \leq n) \}$
$\{(x = (i+1)^3) \wedge ((y+z) = (i+2)^3 - (i+1)^3) \wedge (z = 6i+6) \wedge ((i+1) \leq n) \}$

y := y +z;
$\{(x = (i+1)^3) \wedge (y = (i+2)^3 - (i+1)^3) \wedge (z = 6i+6) \wedge ((i+1) \leq n) \}$
$\{(x = (i+1)^3) \wedge (y = ((i+1)+1)^3 - (i+1)^3) \wedge (z = 6i+6) \wedge ((i+1) \leq n) \}$
$\{(x = (i+1)^3) \wedge (y = ((i+1)+1)^3 - (i+1)^3) \wedge ((z+6) = 6i+12) \wedge ((i+1) \leq n) \}$

z := z + 6;
$\{(x = (i+1)^3) \wedge (y = ((i+1)+1)^3 - (i+1)^3) \wedge (z = 6i+12) \wedge ((i+1) \leq n) \}$
$\{(x = (i+1)^3) \wedge (y = ((i+1)+1)^3 - (i+1)^3) \wedge (z = 6(i+1) + 6) \wedge ((i+1) \leq n) \}$

i := i+1;
$\{(x = i^3) \wedge (y = (i+1)^3 - i^3) \wedge (z = 6i + 6) \wedge (i \leq n) \}$

$\{(x = i^3) \wedge (y = (i+1)^3 - i^3) \wedge (z = 6i + 6) \wedge (i \leq n) \wedge \neg (i < n)\}$
$\{(x = i^3) \wedge (i = n)\}$
$\{ x = n^3 \}$