Propositional Natural Deduction

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Logic

- In CS 60 we had an introduction to both **proposition-** and **predicate-logic**.
- These were covered from the viewpoint of **meaning** (known as “model theory” to logicians).
- There is another part of the story dealing with the structure of **proofs** (known as “proof theory”).
- We focus on the latter now, and will connect the two eventually.

Logic in CS 81

- We have two **objectives** in studying formal logic:
  - To firm up our concept of what forms a proof and how to create proofs.
  - To investigate the connection between computability and provability, such as:
    - The problem of giving an algorithm that will determine whether or not certain kinds of statements can be proved from certain axioms is **unsolvable**.

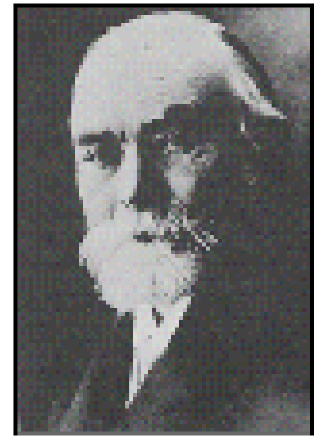
Formal Systems

- A system of logical proof is a variety of **formal system**, just as grammars and Turing machines are formal systems.

- A formal system tells how to **construct** things, using **precise rules**, usually as some form of induction.

- "Formal" means that adherence to the rules can be checked **algorithmically**.

Gottlob Frege (1848-1925)



- Created modern logic by introducing the predicate calculus.
- Developed a formalized definition of “proof”.
- Defined the natural numbers, anticipating Peano’s axiomatization (1889).
- Did not anticipate Russell’s paradox.

Russell's Paradox (1902)

- Consider the "set" P:

$$P = \{S \mid S \text{ is a set and } S \notin S\}$$

- Is $P \in P$?
  - If $P \in P$, then **not** $P \notin P$.
  - If $P \notin P$, then $P \in P$.

- We are forced to conclude that P is **not** a set after all.

Diagonal Arguments

- Russell’s paradox is a typical “diagonal argument”, the first of which was given by Cantor in showing the existence of uncountable sets.

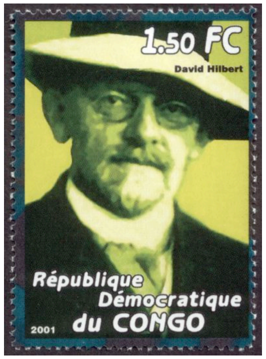
- The same kind of construction is used to show the existence of uncomputable functions.

Varieties of Logical Proof Systems

- Axiomatic or Hilbert/Ackermann:
  - Basis is a set of **axioms**
  - Rules of inference tell how to derive **theorems** from axioms (in zero or more steps).
  - Relatively few rules of inference.
- Natural Deduction, or Gentzen:
  - No axioms
  - Rules of inference tell how to derive **sequents**, which can entail axioms as pre-conditions and theorems as post-conditions.
  - Relatively many rules of inference.
- The two are equivalent; it is a matter of style.

Hilbert/Ackerman and Gentzen

- David Hilbert (1862-1943)



- Wilhelm Ackermann (1896-1962) student of Hilbert

- Gerhard Gentzen (1909-1945)

Natural Deduction (Gentzen)

- A natural deduction system derives **sequents**, expressions of the form:

  $$\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$$

- Each of the φ_i and ψ represents a **logical formula** in an appropriate language. Formulae are also called “statements” or “judgements”.

- The interpretation of the sequent is that each φ_i is a **premise** and ψ is the **conclusion**.

- The left-hand side is a **set**; order is not significant.

- The φ_i *could* be called **axioms**, then ψ would be a **theorem**.

Truth vs. Derivation

- The **intended interpretation** of the sequent $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ is that ψ is a **true** formula provided that each of the φ_i are true.

- Whether or not this is really the case will depend on the rules for constructing sequents.

- The definition of "**truth**" will be given later, but you can assume that it is like the one you know.

- Derivations themselves do not rely on notions of truth; they are totally **mechanical**.

A Propositional Formula Language

- E is the root of the grammar
- E → A | // Atom
  (¬E) | // Negation (not)
  (E ∧ E) | // Conjunction (and)
  (E ∨ E) | // Disjunction (or)
  (E '→' E) | // Implication (implies)
  ⊥ | // Bottom (false)
  T // Top (true)

- A → 'a' | 'b' | … | 'p' | 'q' | 'r' | 's' | … 'z'

Precedence

- The language as given fully parenthesizes everything.
- We will allow use of precedence in lieu of parentheses as an **abbreviation**.
- The binding order is negation, conjunction, disjunction, then implication.
- Grouping is right-to-left.
- So

$$((p \wedge ((\neg q) \wedge r))) \vee (((\neg r) \wedge s) \rightarrow q)))$$

could be abbreviated:

$$(p \wedge \neg q \wedge r) \vee (\neg r \wedge s \rightarrow q)$$

To Clarify

- $p \rightarrow q \rightarrow r$ is $p \rightarrow (q \rightarrow r)$, *not* $(p \rightarrow q) \rightarrow r$
- $p \wedge q \rightarrow r$ is $(p \wedge q) \rightarrow r$, *not* $p \wedge (q \rightarrow r)$
- $p \vee q \rightarrow r$ is $(p \vee q) \rightarrow r$, *not* $p \vee (q \rightarrow r)$

Examples of Sequents

- $p, (p \rightarrow q) \vdash q$

- $(p \vee q), \neg p \vdash q$

- $(p \rightarrow q), (p \vee r), \neg r \vdash q$

- The first, for example, could be interpreted "if p is true and $(p \rightarrow q)$ is true, then q is true".

- Sequents are derived via proofs, to be demonstrated shortly.

Sequents and Intuition

- You might be thinking “Why bother with sequents; I can do all of this with my knowledge of tautologies, etc.”
- Your knowledge can be used as **intuition** for validating a sequent.
- However, sequents express whether certain **deductions** are valid, as they might occur in a mathematical proof.
- Tautologies won’t be enough when we extend the language to include predicates and quantifiers.
- In addition to **using** sequents, we intend to **study** the **proof systems** themselves (called meta-logic).

Natural Deduction Rules

- Each rule represents an **allowable step** in deriving a sequent.
- The rules focus on deriving formulas by **introducing** or **eliminating** the various connectives:
  - $\neg$
  - $\wedge$
  - $\vee$
  - $\rightarrow$
- There is one rule for each case (introduction and elimination) for at least each connective, i.e. at least 8 rules. Some rules have multiple sub-rules.

Why “Natural” Deduction?

- “Natural” is a slogan intending to suggest that these rules are ones that might be used in normal proof construction and argumentation.

- Natural deduction also allows an argument to be developed by examining the desired conclusion and working toward assumed premises in a “natural” way.

Natural Deduction Inference Rules

∧-Introduction Rule (∧i)

- $$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \quad (\wedge i)$$

- The reading of this rule is:
  - If φ and ψ are any formulas that follow from the premises of a sequent,

    then the formula $\varphi \wedge \psi$ also follows from those premises.
- The formulas above the line are called the **antecedents** and the one below the **consequent**.

Examples of Sequents Derived Using Only the (∧i) Rule

- p, (q∨r) ⊢ p∧(q∨r) [One rule app.]
- p, (q∨r) ⊢ (q∨r)∧p [One rule app.]
- p, (q∨r), s ⊢ ((q∨r)∧(p∧s)) [Two rule apps.]

Showing Sequent Derivations by Steps

- Derive p, $(q\vee r)$, s $\vdash$ $((q\vee r)\wedge(p\wedge s))$:

1.	p	Premise
2.	$(q\vee r)$	Premise
3.	s	Premise
4.	$(p\wedge s)$	Rule $\wedge$i applied to formulas 1, 3
5.	$((q\vee r)\wedge(p\wedge s))$	Rule $\wedge$i applied to formulas 2, 4

- The numbers on the right refer to the **antecedents** used in the rule to obtain the formula on the left, which is the **consequent** of a rule.

Rule vs. Sequent

- Every rule immediately justifies an **infinite** number of sequents. For example, the rule

  $$\frac{\varphi \quad \psi}{\varphi \wedge \psi}$$

  creates sequents **of the form**

  $$\varphi, \psi \vdash (\varphi \wedge \psi)$$

  for every pair of formulas φ and ψ.

- The greek letters in the sequent form shown are not **the** formulas; they stand for arbitrary formulas.

- Most sequents require **multiple** rule applications to establish, i.e. proofs of length more than 1.

Showing Sequent Derivations by DAGs

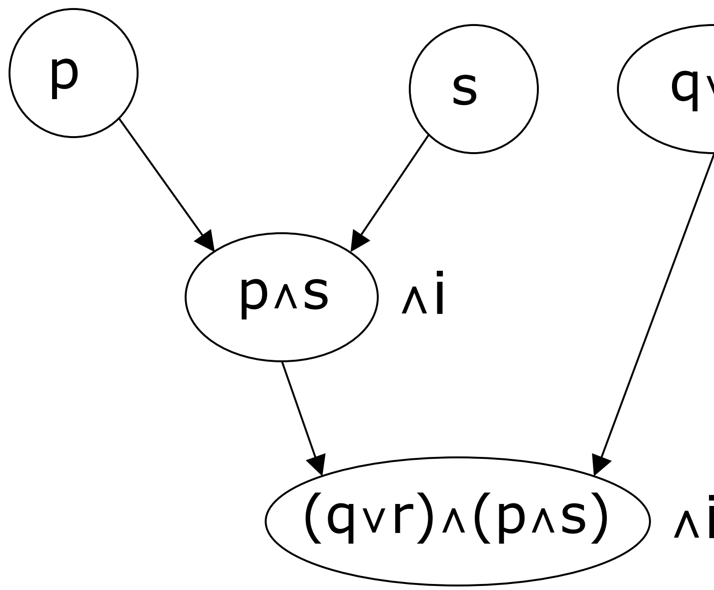
- DAG = "Directed Acyclic Graph"
- The premises are at the *leaves* of the DAG.

$$\dfrac{\dfrac{p \qquad s}{(p\wedge s)}\;\wedge i \qquad (q\vee r)}{((q\vee r)\wedge(p\wedge s))}\;\wedge i$$

- Note that $(p\wedge s)$ is used as the consequent of one rule application and the antecedent of another.

DAG made more evident

$$\dfrac{\dfrac{p \quad s}{(p \wedge s)}\ \wedge i \qquad (q \vee r)}{((q \vee r) \wedge (p \wedge s))}\ \wedge i$$



Steps vs. DAGs

- Steps correspond to the way that an argument might be presented in a math text or paper.
- DAGs allow for better visualization of what is used for what.
- Either representation can be constructed from the other.

∧-Elimination Rules ($\wedge e_1, \wedge e_2$)

- $$\frac{\varphi \wedge \psi}{\varphi} \qquad (\wedge e_1)$$
- $$\frac{\varphi \wedge \psi}{\psi} \qquad (\wedge e_2)$$
- Two sub-rules are needed because **order matters *within*** a formula. This rule eliminates one side of the ∧ or the other.

A Step Derivation Using $\wedge$e and $\wedge$i

- Derive $p \wedge (q \wedge r) \vdash (p \wedge q) \wedge r$:

1.	$p \wedge (q \wedge r)$	Premise
2.	p	$\wedge e_1$ 1
3.	$q \wedge r$	$\wedge e_2$ 1
4.	q	$\wedge e_1$ 3
5.	r	$\wedge e_1$ 3
6.	$p \wedge q$	$\wedge i$ 2, 4
7.	$(p \wedge q) \wedge r$	$\wedge i$ 6, 5

A DAG Derivation Using $\wedge$e and $\wedge$i

- Derive $p \wedge (q \wedge r) \vdash (p \wedge q) \wedge r$:

$\underline{p \wedge (q \wedge r)}$ $\wedge e_1$ $\wedge e_2$

p $\underline{(q \wedge r)}$ $\wedge e_1$ $\wedge e_2$

$\underline{q}$ r $\wedge i$

$\underline{(p \wedge q)}$ $\wedge i$

$(p \wedge q) \wedge r$

This shows that the DAG is not generally a “tree”, as some antecedents are used multiple times.

Every DAG could be made into a tree by copying the shared antecedents.

DAG made more evident

- Derive p ∧ (q ∧ r) ⊢ (p ∧ q) ∧ r :

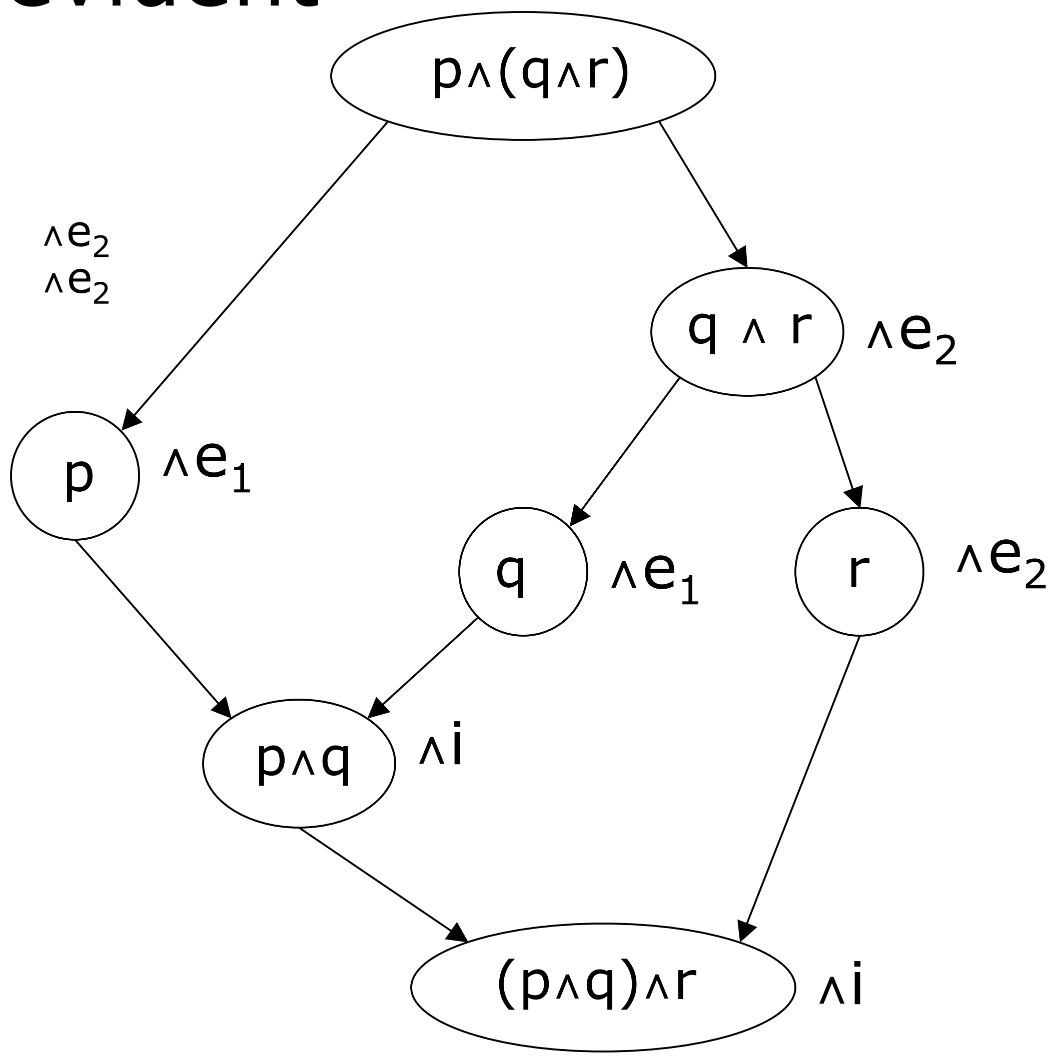
p ∧ (q ∧ r) ∧e$_1$ ∧e$_2$

p (q ∧ r) ∧e$_1$ ∧e$_2$

q r ∧i

(p ∧ q) ∧i

(p ∧ q) ∧ r



Constructing Proofs by Working Backward

- If the conclusion is a premise, there is nothing to do.
- Otherwise, the **outermost** logical connective may suggest what rule could be used:
  - Derive $p \wedge (q \wedge r) \vdash (p \wedge q) \wedge r$
  - The outermost connective in the conclusion is $\wedge$ therefore use $\wedge$i as the last step:
    - $(p \wedge q) \wedge r$ $\wedge$i 6, 5
  - The use of $\wedge$i would require derivation of two new formulas:
    - $(p \wedge q)$ r
  - Apply this approach recursively.

Constructing Proofs by Working Forward

- If a premise is the conclusion, there is nothing to do.
- Otherwise, synthesize a formula from existing formulas using available rules.
- Working forward might entail many choices of a formula to be synthesized, not all of which will be useable in deriving the conclusion.

Choices

- Often the rule choice is not unique.
- Make a choice, but be prepared to backtrack (discarding some of what you have done) and try a different one.

Constructing Proofs by Working Both Directions Simultaneously

- Blend together working backward with working forward until the two “meet in the middle”.
- Don’t overlook the DAG model as a means of arriving at or clarifying proofs.
- Consider converting the DAG to steps for final presentation.

∨-Introduction Rules ($\vee i_1$, $\vee i_2$)

- $$\frac{\varphi}{\varphi \vee \psi} \quad (\vee i_1)$$

- $$\frac{\psi}{\varphi \vee \psi} \quad (\vee i_2)$$

$\rightarrow$-Elimination Rule, Modus Ponens

- $$\frac{\varphi \qquad \varphi \rightarrow \psi}{\psi} \qquad (\rightarrow e)$$

- Its latin name ***modus ponens*** (**MP**), "method of affirming", is often used for this rule.

Example using $\rightarrow$-Elimination Rule

- Derive p, (p $\rightarrow$ q), (q $\rightarrow$ r) $\vdash$ r

1.	p	Premise
2.	p $\rightarrow$ q	Premise
3.	q $\rightarrow$ r	Premise
4.	q	$\rightarrow$e 1, 2
5.	r	$\rightarrow$e 4, 3

- With this example, you can start to see how deriving a sequent might actually be easier (and more “natural”) than using truth tables, etc.

Another form of →-Elimination Rule, Modus Tollens

- A related "derived rule" is ***modus tollens*** (**MT**), "method of denying":

- $$\frac{\varphi \rightarrow \psi \qquad \neg\psi}{\neg\varphi} \qquad \text{(MT)}$$

- This could be called a "macro" rule, meaning that this rule is a convenience and can be treated as an abbreviation for the application of other rules.
- We will justify this shortly.

Example using MT

- Derive $\neg r, (p \rightarrow q), (q \rightarrow r) \vdash \neg p$

1.	$\neg r$	Premise
2.	$p \rightarrow q$	Premise
3.	$q \rightarrow r$	Premise
4.	$\neg q$	MT 3, 1
5.	$\neg p$	MT 2, 4

¬¬-Elimination and Introduction Rules

- $$\frac{\neg\neg\varphi}{\varphi}$$ (¬¬e)

- $$\frac{\varphi}{\neg\neg\varphi}$$ (¬¬i) (This rule is "derived".)

Rules with Sub-Derivations

- Certain rules have **sub-derivations**, rather than simply formulas, in their **antecedents**.
- A sub-derivation may incorporate **assumptions** that behave as **premises** but are not premises of the sequent being proved.
- These assumptions must be treated carefully to avoid confusion with regular premises.
- Accordingly, sub-derivations are shown inside a **box**.
- **Assumptions introduced inside the box cannot be used as premises outside the box.**
- **However**, sub-derivations **may** use formulas derived earlier **outside** the box.

φ

.

.

.

ψ

→-Introduction Rule

- This is an example of a rule using a sub-derivation.

$$\frac{\boxed{\begin{array}{c}\varphi \\ \cdot \\ \cdot \\ \cdot \\ \psi\end{array}}}{\varphi \to \psi} \quad (\to i)$$

- Here to derive we use φ as an **assumption** and get ψ as a conclusion using a sub-derivation.
- φ and ψ alone are not useable outside the box.

Example Using Sub-Derivation

- Derive $(p \rightarrow q), (q \rightarrow r) \vdash (p \rightarrow r)$

1.	$p \rightarrow q$	Premise	
2.	$q \rightarrow r$	Premise	
3.	p	Assumption	
4.	q	$\rightarrow$e	1, 2
5.	r	$\rightarrow$e	2, 4
6.	$p \rightarrow r$	$\rightarrow$i	2-5

Another Example Using Sub-Derivation

- Derive $(\neg p \rightarrow \neg q) \vdash (q \rightarrow p)$:

1.	$\neg p \rightarrow \neg q$	Premise	
2.	q	Assumption	
3.	$\neg\neg q$	$\neg\neg$i	2
4.	$\neg\neg p$	MT	1, 3
5.	p	$\neg\neg$e	4
6.	$q \rightarrow p$	$\rightarrow$i	2-5

- Pattern matching:

  $\dfrac{\varphi\rightarrow\psi,\ \neg\psi}{\neg\varphi}$ (MT) φ is $\neg p$, ψ is $\neg q$,
  $\neg\varphi$ is $\neg\neg p$, $\neg\psi$ is $\neg\neg q$

A Sub-Derivation can be Trivial

- Derive $\vdash (p \rightarrow p)$ (Set of premises is empty):

1.	p	Assumption	
2.	$p \rightarrow p$	$\rightarrow$i	1, 1

- Pattern matching:

$$\frac{\boxed{\begin{array}{c}\varphi \\ \vdots \\ \psi\end{array}}}{\varphi \rightarrow \psi} \quad (\rightarrow i)$$

Both φ and ψ are p.

Sub-Derivations can be Nested

- Derive $(p \wedge q) \rightarrow r \vdash p \rightarrow (q \rightarrow r)$

1.	$(p \wedge q) \rightarrow r$	Premise
2.	p	Assumption
3.	q	Assumption
4.	$p \wedge q$	$\wedge$i 2,3
5.	r	$\rightarrow$e 1, 4
6.	$q \rightarrow r$	$\rightarrow$i 3-5
7.	$p \rightarrow (q \rightarrow r)$	$\rightarrow$i 2-6

Try These:

- $p \rightarrow (q \rightarrow r) \vdash (p \wedge q) \rightarrow r$
- $p \rightarrow (q \wedge r) \vdash (p \rightarrow q) \wedge (p \rightarrow r)$

∨-Elimination Rule

- This rule uses two sub-derivations:

$$\frac{\varphi \vee \psi \quad \boxed{\begin{array}{c}\varphi\\ \vdots\\ \chi\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}\ (\vee e)$$

- The interpretation is that if we want to "get rid of" a disjunction, we can derive a common formula χ from the two disjuncts φ and ψ.

Example

- $p \vee q \vdash q \vee p$

1.	$p \vee q$	Premise
2.	p	Assumption
3.	$q \vee p$	$\vee i_2$ 2
4.	q	Assumption
5.	$q \vee p$	$\vee i_1$ 4
6.	$q \vee p$	$\vee e$ 1, 2-3, 4-6

Try These:

- $(p \rightarrow q) \vee (p \rightarrow r) \vdash p \rightarrow (q \vee r)$

- $(p \rightarrow q) \vdash (p \vee r) \rightarrow (q \vee r)$

Sub-Derivations vs. Sequents?

- Aren't the boxed sub-derivations essentially sequents themselves, and
- if so, why don't we use the notation $\varphi \vdash \psi$ rather than

$$\boxed{\begin{array}{c}\varphi\\ \cdot\\ \cdot\\ \cdot\\ \psi\end{array}}$$

- The answer lies in the fact that sub-derivations can make use of formulas **outside** the box, and we'd have to repeat those formulas as premises of the sequent, which would be clumsy.

¬-Introduction Rule

This rule introduces ¬ through "contradiction":

$$\frac{\boxed{\begin{matrix}\varphi \\ \vdots \\ \bot\end{matrix}}}{\neg\varphi}\ (\neg i)$$

¬-Elimination Rule

$$\frac{\varphi \qquad \neg\varphi}{\bot} \quad (\neg e)$$

⊥-Elimination Rule

$$\frac{\bot}{\varphi} \quad (\bot e)$$

- If we can derive ⊥ (false) then we can derive anything. Consequently, the things we derive won't have much information value.
- So being able to derive ⊥ is undesirable, **except in a sub-derivation**.

Macro or Derived Rules

- Earlier MT was mentioned as a “macro” rule.
- The name “macro” alludes to programming language macros, which expand textually.
- While superficially similar to a subroutine, a macro is a text substitution done before a source is compiled or interpreted.
- In our case, it is a rule that could be replaced with a sequence of uses of other rules.

Deriving MT as a Macro from other rules

- $\dfrac{\varphi \rightarrow \psi, \ \neg \psi}{\neg \varphi}$ (MT)

1.	$\varphi \rightarrow \psi$	Premise
2.	$\neg \psi$	Premise
3.	φ	Assumption
4.	ψ	$\rightarrow$e 1, 3
5.	$\bot$	$\neg$e 4, 2
6.	$\neg \varphi$	$\neg$i 3-5

- Every use of MT could thus be replaced with these steps, which use 3 rules: $\rightarrow$e, $\neg$e, $\neg$i.

$\neg\neg$i as a Macro derived from other rules

- $\dfrac{\varphi}{\neg\neg\varphi}$ ($\neg\neg$i)

1.	φ	Premise
2.	$\neg\varphi$	Assumption
3.	$\bot$	$\neg$e 1, 2
4.	$\neg\neg\varphi$	$\neg$i 2-3

Macro vs. Sequent

- Why isn't a macro rule just another sequent?

RAA (Reductio ad absurdum) Rule

- Also known as PBC (Proof by Contradiction)
- This rule has a similarity to $\neg i$, but it is different:

$$\frac{\boxed{\begin{array}{c} \neg\varphi \\ \vdots \\ \bot \end{array}}}{\varphi}\ \text{(RAA)}$$

Review of ¬i and ¬e

$$\frac{\boxed{\begin{array}{c}\varphi \\ \vdots \\ \bot\end{array}}}{\neg\varphi}\ (\neg i)$$

$$\frac{\varphi \wedge \neg\varphi}{\bot}\ (\neg e)$$

RAA as a Macro derived from other rules

1.	$\neg\varphi$ . . . $\bot$	Premise
2.	$\neg\varphi \to \bot$	$\to$i 1
3.	$\neg\varphi$	Assumption
4.	$\bot$	$\to$e 2,3
5.	$\neg\neg\varphi$	$\neg$i 3-4
6.	φ	$\neg\neg$e 5

LEM (Law of the Excluded Middle) Derived

- $\dfrac{\qquad}{\varphi \lor \neg\varphi}$ (Empty antecedent)

1.	$\neg(\varphi \lor \neg\varphi)$	Assumption
2.	φ	Assumption
3.	$\varphi \lor \neg\varphi$	$\lor i_1$ 2
4.	$\bot$	$\neg e$ 3, 1
5.	$\neg\varphi$	$\neg i$ 2-4
6.	$\varphi \lor \neg\varphi$	$\lor i_2$ 2
7.	$\bot$	$\neg e$ 6, 1
8.	$\neg\neg(\varphi \lor \neg\varphi)$	$\neg i$ 1, 7
9.	$\varphi \lor \neg\varphi$	$\neg\neg e$ 8

Summary of Intro & Elim Rules

Connective	Introduction	Elimination
$\wedge$	$\wedge$i	$\wedge e_1$, $\wedge e_2$
$\vee$	$\vee i_1$, $\vee i_2$	$\vee$e
$\rightarrow$	$\rightarrow$i	$\rightarrow$e
$\neg$	$\neg$i	$\neg$e
$\bot$	(none)	$\bot$e
$\neg\neg$	$\neg\neg$i (derived)	$\neg\neg$e

Summary of Intro & Elim Rules

Connective	Introduction	Elimination
$\wedge$	$\dfrac{\varphi \quad \psi}{\varphi\wedge\psi}$ ($\wedge$i)	$\dfrac{\varphi\wedge\psi}{\psi}$ ($\wedge e_1$) $\dfrac{\varphi\wedge\psi}{\varphi}$ ($\wedge e_2$)
$\vee$	$\dfrac{\varphi}{\varphi \vee \psi}$ ($\vee i_1$) $\dfrac{\psi}{\varphi \vee \psi}$ ($\vee i_2$)	$\dfrac{\varphi \vee \psi \quad \boxed{\varphi \ldots \chi} \quad \boxed{\psi \ldots \chi}}{\chi}$ ($\vee$e)
$\rightarrow$	$\dfrac{\boxed{\varphi \ldots \psi}}{\varphi \rightarrow \psi}$ ($\rightarrow$ i)	$\dfrac{\varphi \quad \varphi\rightarrow\psi}{\psi}$ ($\rightarrow$e or MP)
$\neg$	$\dfrac{\boxed{\varphi \ldots \bot}}{\neg\varphi}$ ($\neg$i)	$\dfrac{\varphi \quad \neg\varphi}{\bot}$ ($\neg$e)
$\bot$	(none)	$\dfrac{\bot}{\varphi}$ ($\bot$e)
$\neg\neg$	$\dfrac{\varphi}{\neg\neg\varphi}$ ($\neg\neg$i) (derived)	$\dfrac{\neg\neg\varphi}{\varphi}$ ($\neg\neg$e)

Summary of *Derived* Rules So Far

- MT (Modus Tollens)
- RAA (Reductio ad Absurdum)
- LEM (Law of the Excluded Middle)
- ¬¬i

Intuitionistic Logic

- We have presented classical natural deduction rules.
- Of these, certain rules are suspect by a school of thought known as “intuitionism”.
- Intuitionism rejects:
  - LEM
  - RAA
  - $\neg\neg e$

Why Intuitionism rejects LEM

- Intuitionism says that in order to prove a statement $\varphi \vee \neg\varphi$, one must be able to construct explicitly which of φ vs. $\neg\varphi$ is true.

- "proofs" that use LEM in an essential way would be rejected.

Example of Non-Constructive LEM Usage

- "There are two irrational numbers a and b such that a^b is rational."
- "Proof: (We assume that it has been established that $\sqrt{2}$ is irrational).
- The number $(\sqrt{2})^{\sqrt{2}}$ is either rational or it isn't (LEM).
- If $(\sqrt{2})^{\sqrt{2}}$ is rational, take $a = b = \sqrt{2}$ and we are done.
- If $(\sqrt{2})^{\sqrt{2}}$ is not rational, take $a = (\sqrt{2})^{\sqrt{2}}$ and $b = \sqrt{2}$, for then $a^b = ((\sqrt{2})^{\sqrt{2}})^{\sqrt{2}} =$
  $(\sqrt{2})^{\sqrt{2}\sqrt{2}} = (\sqrt{2})^2 = 2$ and we are done."
- This proof of the existence of something without constructing it is suspect in intuitionism.

Proof that $\sqrt{2}$ is irrational

- Suppose that $\sqrt{2}$ is rational.
- By the definition of a rational number, there are integers p, q such that $\sqrt{2} = p/q$, with p and q having no common factors.

- Since p and q have no common factors, neither can their squares.

- But squaring both sides, $2 = p^2/q^2$, indicating that all factors of q^2 are factors of p^2.

- This is a contradiction, so $\sqrt{2}$ is irrational.

Why Intuitionism rejects ¬¬e

- With ¬¬e and other non-controversial rules, LEM can be derived, as we have seen.
- Therefore rejecting LEM means we must reject ¬¬e.
- Alternatively, ¬¬e could be derived from RAA without using ¬¬e explicitly, so RAA must also be rejected.

Derive ¬¬e using RAA

- $\dfrac{\neg\neg\varphi}{\varphi}$ (¬¬e)
- Proof:

1.	$\neg\neg\varphi$	Premise
2.	$\neg\varphi$	Assumption
3.	$\bot$	$\bot$i 1, 2
4.	φ	RAA 2-3

The Rules in the Hein Text

- The Hein Text has some slightly different rules (p 370-371).
  - MP (6.7) is one of our rules.
  - MT (6.8) is one of our derived rules.
  - Conjunction (6.9) is $\wedge$i.
  - Simplification (6.10) is $\wedge e_1$.
  - Addition (6.11) is $\vee i_1$.
  - (continued next slide)

The Rules in the Hein Text, cont'd.

- DS (Disjunctive Syllogism) (6.12)

$$\frac{\varphi \vee \psi,\ \neg\varphi}{\psi} \quad \text{(DS)}$$

- Derivation:

1.	$\varphi \vee \psi$	Premise
2.	$\neg\varphi$	Premise
3.	φ	Assumption
4.	$\bot$	$\neg$e 2, 3
5.	ψ	$\bot$e 2-4
6.	ψ	Assumption
7.	ψ	$\vee$e 1, 3-6, 6-6

The Rules in the Hein Text, cont'd.

- HS (Hypothetic Syllogism) (6.13)

$$\frac{\varphi \rightarrow \psi,\ \psi \rightarrow \chi}{\varphi \rightarrow \chi} \quad \text{(HS)}$$

- We derived a sequent of this form earlier, so just redo that proof for the more general formulas.

The Rules in the Hein Text, cont'd.

- CD (Constructive Dilemma) (6.14)

$$\frac{\varphi \vee \chi,\ \varphi \rightarrow \psi,\ \chi \rightarrow \xi}{\psi \vee \xi} \qquad \text{(CD)}$$

1.	$\varphi \vee \chi$	Premise
2.	$\varphi \rightarrow \psi$	Premise
3.	$\chi \rightarrow \xi$	Premise
4.	φ	Assumption
5.	ψ	MP 2, 4
6.	$\psi \vee \xi$	$\vee i_1$ 5
7.	χ	Assumption
8.	ξ	MP 3, 5
9.	$\psi \vee \xi$	$\vee i_2$ 8
10.	$\psi \vee \xi$	$\vee e$ 1, 4-6, 7-9

The Rules in the Hein Text, cont'd.

- DD (Destructive Dilemma) (6.14)

$$\frac{\neg\varphi \lor \neg\chi,\ \psi \to \varphi,\ \xi \to \chi}{\neg\psi \lor \neg\xi} \quad \text{(DD)}$$

1.	$\neg\varphi \lor \neg\chi$	Premise
2.	$\psi \to \varphi$	Premise
3.	$\xi \to \chi$	Premise
4.	$\neg\varphi$	Assumption
5.	$\neg\psi$	MT 2, 4
6.	$\neg\psi \lor \neg\xi$	$\lor i_1$ 5
7.	$\neg\chi$	Assumption
8.	$\neg\xi$	MT 3, 5
9.	$\neg\psi \lor \neg\xi$	$\lor i_2$ 8
10.	$\neg\psi \lor \neg\xi$	$\lor e$ 1, 4-6, 7-9

Other Rules in Hein

- What about RAA, etc?
  Hein introduces IP (Indirect Proof Rule)(6.17) on p. 378, which is essentially RAA.
- In the same rule, he also introduces, in effect:

$$\frac{\boxed{\begin{array}{c} \varphi \wedge \neg\psi \\ \vdots \\ \bot \end{array}}}{\varphi \rightarrow \psi} \quad \text{(IP)}$$

Proofs in Hein Book

- The proofs are similar to the style we have shown.

- Hein uses indenting, but **no boxes**, which is slightly less clear.

- NOTE: Hein in places uses “simplifications” by substituting one formula for another “equivalent” formula. Until some additional results have been shown, we will **not** allow this, since there is no clear rule for it.

Proofs in Hein Book

- So the natural deduction proofs in Hein are a bit of a hodge-podge of formal rules and informal ones.
- For now, your proofs should abide by the strict principles that we have laid out, using the natural deduction rules as a foundation.
- Do not use logical equivalences and the like at this time.

Hein Section 6.4

- Here Hein introduces three axiom-based deduction systems:
  - Frege-Lukasiewicz
  - Frege
  - Hilbert-Ackerman

- As mentioned before, these have several axioms, but usually only MP as the rule of inference.

- They should be equivalent to the natural deduction system, but this itself would need to be proved (by showing by induction that anything provable in one system is provable in the other).

Other References

- Michael Huth and Mark Ryan, *Logic in Computer Science*, 2nd Edition, Cambridge University Press, 2004 (out of print). [I used the rules as stated there.]

- Dirk van Dalen, *Logic and Structure*, 3rd Augmented Edition, Springer-Verlag, 1997 [More rigorous, fewer rules and connectives by defining some connectives in terms of others.]