Number Theory in Logic

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9 February 2005

Number Theory

- By “number theory” we mean the simple theory of the **natural numbers** that includes the addition and multiplication functions (in contrast to Peano arithmetic, which does not include these functions).
- This interpretation is known as the “standard model” for number theory.

Axioms of Number Theory (N)

N1: $(\forall x)\ \neg(S(x) = 0)$

N2: $(\forall x)\ (\forall y)\ ((S(x) = S(y)) \rightarrow (x = y))$

N3: $(\forall x)\ (x + 0) = x$

N4: $(\forall x)\ (\forall y)\ (x + S(y)) = S(x+y)$

N5: $(\forall x)\ (x * 0) = 0$

N6: $(\forall x)\ (\forall y)\ (\forall z)\ (x*S(y)) = ((x*y) + x)$

N7: $(\varphi[0/x] \wedge (\forall x)\ (\varphi \rightarrow \varphi[S(x)/x])) \rightarrow (\forall x)\varphi$ for any formula φ

Comment on Axioms

- S is the successor (“add 1”) function.
- N1 and N2 are from the Peano axioms.
- N3 and N4 are key properties of +; they are like a recursive definition.
- N5 and N6 are similarly key properties of * (multiplication).

Comment on Induction Axiom

- N7 is the principle of mathematical induction, which is also one of the Peano axioms.
- Note that N7 is really an infinite set of axioms, as there are infinitely-many choices for φ.

More Comments on the Axioms

- Notice that familiar identities such as the associative and commutative laws of additiona and multiplication are absent.

- This is intentional: these laws can be **derived** from the axioms using induction.

Induction as a Rule

- **N7**: $(\varphi[0/x] \wedge (\forall x)(\varphi \rightarrow \varphi[S(x)/x])) \rightarrow (\forall x)\varphi$

- To reduce the number of rule uses (of $\wedge$i and $\rightarrow$ i), we are going to use a **derived inference rule** in place of N7 most of the time:

- **Induc**:

$$\frac{\varphi[0/x] \qquad \boxed{\begin{array}{l}\varphi \\ \ldots \\ \varphi[S(x)/x]\end{array}}}{(\forall x)\varphi}$$

for any formula φ and any variable x

Theorem 1: $N \vdash (\forall x)\ (0 + x) = x$

Basis

1.	$(\forall x)\ (x + 0) = x$	N3
2.	$0 + 0 = 0$	$\forall$e 1

Induction Step

3.	$0 + x = x$	Assumption
4.	$S(0 + x) = S(x)$	= subst.
5.	$(\forall x)\ (\forall y)\ (x + S(y)) = S(x+y)$	N4
6.	$(0 + S(x)) = S(0 + x)$	$\forall$e (twice) 5
7.	$(0 + S(x)) = S(x)$	= 6, 4

Conclusion

8.	$(\forall x)\ (0 + x) = x$	Induc. 2, 3-7

Theorem 2: $N \vdash (\forall y)(\forall x)(S(x) + y) = S(x + y)$

1.	$(\forall x)(x + 0) = x$	N3
2.	$S(x) + 0 = S(x)$	$\forall$e 1
3.	$x + 0 = x$	$\forall$e 1
4.	$S(x) + 0 = S(x+0)$	= 3, 2
5.	$(\forall x)\ S(x) + y = S(x + y)$	Assumption
6.	x	
7.	$S(x) + y = S(x + y)$	$\forall$e 5
8.	$(\forall x)(\forall y)(x + S(y)) = S(x+y)$	N4
9.	$S(x) + S(y) = S(S(x) + y)$	$\forall$e 8 (twice)
10.	$S(x) + S(y) = S(S(x + y))$	= 7, 9
11.	$(\forall x)(\forall y)(x + S(y)) = S(x+y)$	N4
12.	$x + S(y) = S(x+y)$	$\forall$e 11 (twice)
13.	$S(x) + S(y) = S(x + S(y))$	= 10, 12
14.	$(\forall x)\ S(x) + S(y) = S(x + S(y))$	$\forall$i 6-13
15.	$(\forall y)(\forall x)(S(x) + y) = S(x + y)$	Induc 4, 5-14

Theorem 3: N ⊢ $(\forall x)(\forall y)\ (x + y) = (y + x)$

Basis

1.	y	
2.	$(\forall x)\ (x + 0) = x$	N3
3.	$y + 0 = y$	∀e 2
4.	$(\forall x)\ (0 + x) = x$	Theorem 1
5.	$0 + y = y$	∀e 4
6.	$0 + y = y + 0$	= 3, 5
7.	$(\forall y)\ 0 + y = y + 0$	∀i 1-6

Theorem 3 con'td: N $\vdash$ $(\forall x)(\forall y)\ (x + y) = (y + x)$

Induction Step

8.	$(\forall y)\ (x + y) = (y + x)$	Assumption
9.	y	
10.	$(x + y) = (y + x)$	$\forall$e 8
11.	$(\forall y)\ (\forall x)\ (S(x) + y) = S(x + y)$	Theorem 2
12.	$(S(x) + y) = S(x + y)$	$\forall$e 11 (twice)
13.	$(S(x) + y) = S(y + x)$	= 10, 12
14.	$(\forall x)\ (\forall y)\ (x + S(y)) = S(x+y)$	N4
15.	$(y + S(x)) = S(y+x)$	$\forall$e 15 (twice ***)
16.	$(S(x) + y) = (y + S(x))$	= 13, 15
17.	$(\forall y)\ (S(x) + y) = (y + S(x))$	$\forall$i 9-16

Conclusion

18. $(\forall x)(\forall y)\ (x + y) = (y + x)$ Induc 7, 8-17

*** see next page

Note on Double Substitution

On the previous page we indicated

15.	$(\forall x)\ (\forall y)\ (x + S(y)) = S(x+y)$	N4
16.	$(y + S(x)) = S(y+x)$	$\forall$e 15 (twice ***)

This doesn't quite work sequentially, for y is not free to replace x in 8 (y would become bound).

Instead, we'd have substitute a third variable, say z, first, then substitute x for y, then regeneralize z, then substitute x for z, as in:

15.	$(\forall x)\ (\forall y)\ (x + S(y)) = S(x+y)$	N4
16.	z	
17.	$(\forall y)\ (z + S(y)) = S(z+y)$	$\forall$e 15
18.	$(z + S(x)) = S(z+x)$	$\forall$e 17
19.	$(\forall z)\ (z + S(x)) = S(z+x)$	$\forall$i 16-18
20.	$(y + S(x)) = S(y+x)$	$\forall$e 19