Predicate Calculus Semantics

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Syntax vs. Semantics

- Predicate logic proofs, in a system such as natural deduction, focus on **syntax**: each formula in the derivation is **mechanically-checkable** to be derivable from earlier formulas using only the given rules.

- The **semantics** or **meaning** of a formula is determined by separate considerations. Each formula is making a statement about some kind of **underlying mathematical structure**.

Interpretations of Formulas

- The structure(s) of interest in specific derivations are generally **not totally specified** in the system of derivation itself.
- Instead, we rely on certain formulas (“axioms”) to **characterize** the properties of these structures that are of interest. In natural deduction, these formulas will appear on the left-hand side of a sequent.
- It can then be proved separately that the syntactic rules are in agreement with the semantics of the intended **interpretation**.

What is an "Interpretation"?

- An **interpretation** for a set of formulas consists of:
  - A **domain**: that contains all individuals of interest.
  - A mapping from **constant symbols** in the formulas to **specific domain elements** ("constants").
  - A mapping from the **function symbols** in the language to **functions** of corresponding arity mapping n-tuples of domain elements into the.
  - A mapping from the **predicate symbols** in the language to functions of corresponding arity mapping n-tuples of domain elements into {T, F}.

Interpretation (Δ, μ)

- An **interpretation** for a set of formulas consists of:
  - A **domain** Δ: that contains all individuals of interest.
  - For each **constant symbol** c, an element $\mu(c) \in \Delta$**.**
  - For each **function symbol** f, a function $\mu(f)$: $\Delta^n \rightarrow \Delta$.
  - For each **predicate symbol** p, a function $\mu(p)$: $\Delta^n \rightarrow \{T, F\}$.

Assignments for the Predicate Calculus

- Suppose (Δ, μ) is an interpretation for a set Γ of formulas.

- An **assignment** for the interpretation is mapping from the collective *free* variables in Γ to Δ:

$$\alpha\text{: free}(\Gamma) \rightarrow \Delta$$

Example

- Γ = {p(x), p(f(x)), q(c, y), q(f(f(x)), x)}
- Δ = {0, 1, 2}
- μ(c) = 0
- μ(f) = {0 $\rightarrow$ 2, 1 $\rightarrow$ 0, 2 $\rightarrow$ 1}
- μ(p) = {(2)}
  [the set of 1-tuples for which μ(p) is T]
- μ(q) = {(0, 0), (0, 1), (0, 2)}
  [the set of 2-tuples for which μ(q) is T]
- Some assignments α are:
  - {x $\rightarrow$ 0, y $\rightarrow$ 0}
  - {x $\rightarrow$ 0, y $\rightarrow$ 1}
  - {x $\rightarrow$ 1, y $\rightarrow$ 2}, etc. [How many altogether?]

An assignment α for interpretation (Δ, μ) maps the set of **all** terms to a value in Δ.

- Let α be an assignment.
- If t is a variable symbol ν,
  then $\alpha(\nu)$ is just the value to which ν maps by the interpretation.
- If t is a constant symbol c,
  then $\alpha(c) = \mu(c)$, the value assigned by the interpretation.
- If t is $f(t_1, \ldots, t_n)$
  then $\alpha(t) = \mu(f)(\alpha(t_1), \ldots, \alpha(t_n))$.

An assignment α for interpretation (Δ, μ) maps the set of **all** terms to a value in Δ.

- $\Gamma = \{p(x), p(f(x)), q(c, y), q(f(f(x)), x)\}$

- The terms of interest here are:

  x, f(x), c, y, f(f(x))

- Consider the previous interpetation in which $\Delta = \{0, 1, 2\}$, $\mu(c) = 0$, $\mu(f) = \{0 \rightarrow 2, 1 \rightarrow 0, 2 \rightarrow 1\}$, what are the induced values for assignment

  $\{x \rightarrow 0, y \rightarrow 0\}$?

An assignment α for interpretation (Δ, μ) maps the set of **all** terms to a value in Δ.

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  $\{x \rightarrow 1, y \rightarrow 2\}$?

An assignment α for interpretation (Δ, μ) maps every **atomic formula** to {T, F}.

- An atomic formula has the form $p(t_1, \ldots, t_n)$ where p is a predicate symbol and are terms.
- We already defined $\alpha(t_i)$ for terms t_i.
- $\alpha(p(t_1, \ldots, t_n))$ is defined as $\mu(p)(\alpha(t_1), \ldots, \alpha(t_n))$.
- Note: An atomic formula might have **no** free variables. Then the value of any assignment is determined entirely by the interpretation itself.
- Note: An assignment mapping the **empty set** of variable symbols has the value **constant** T or constant F, as determined by the interpretation.

An assignment α for interpretation (Δ, μ) maps the set of *atomic* formulas to {T, F}.

- Γ = {p(x), p(f(x)), q(c, y), q(f(f(x)), x)}
- Consider the previous interpetation in which
  Δ = {0, 1, 2}, μ(c) = 0,
  μ(f) = {0 $\rightarrow$ 2, 1 $\rightarrow$ 0, 2 $\rightarrow$ 1},
  μ(p) = {(2)}
  μ(q) = {(0, 0), (0, 1), (0, 2)}
- What are the values for the atomic formulas for the assignment
  {x $\rightarrow$ 0, y $\rightarrow$ 0}?

An assignment α for interpretation (Δ, μ) maps the set of *atomic* formulas to {T, F}.

- Γ = {p(x), p(f(x)), q(c, y), q(f(f(x)), x)}

- Consider the previous interpetation in which
  Δ = {0, 1, 2}, μ(c) = 0,
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  μ(q) = {(0, 0), (0, 1), (0, 2)}

- What are the values for the atomic formulas for the assignment
  {x $\rightarrow$ 1, y $\rightarrow$ 2}?

An assignment α for interpretation (Δ, μ) maps the set of *quantifier-free* formulas to {T, F}.

- Quantifier-free formulas are just those built from atomic formulas using the propositional connectives $\neg \wedge \vee \rightarrow$.
- Since we know the induced values for atomic formulas, we get the induced values for any quantifier-free formula **using the same definitions as for propositional calculus**.

Assignments

- For any formula of the form $(\varphi \vee \psi)$:

  $\alpha(\varphi \vee \psi) = T$ iff $\alpha(\varphi) = T$ or $\alpha(\psi) = T$.

- For any formula of the form $(\varphi \wedge \psi)$:

  $\alpha(\varphi \wedge \psi) = T$ iff $\alpha(\varphi) = T$ and $\alpha(\psi) = T$.

- For any formula of the form $(\varphi \rightarrow \psi)$:

  $\alpha(\varphi \rightarrow \psi) = T$ iff $\alpha(\varphi) = F$ or $\alpha(\psi) = T$.

- For any formula of the form $(\neg\varphi)$:

  $\alpha(\neg\varphi) = T$ iff $\alpha(\varphi) = F$.

An assignment α for interpretation (Δ, μ) maps the set of **all formulas** to {T, F}.

- Since we know the induced values for **quantifier-free** formulas, we need to show how the values for formulas in general are defined.
- In short, we need to show what happens when the formula contains quantifiers.
- Then we again use propositional rules to determine the value for arbitrary formulas (since these can be formed by combining non-quantifier-free formulas).

Induced value for formulas $(\forall x)\psi$

- Suppose the formula in question is of the form $(\forall x)\psi$.
- We know that for any assignment α we have a truth value $\alpha(\psi) \in \{T, F\}$.

- Furthermore, those assignments also qualify as assignments for $(\forall x)\psi$.

- We define $\alpha((\forall x)\psi)$ to be T provided that **every** assignment α' for ψ that **agrees with** α on free($(\forall x)\psi$) is such that $\alpha'(\psi) = T$. Otherwise the value $\alpha((\forall x)\psi)$ is F.

An assignment α for interpretation (Δ, μ) maps the set of **all formulas** to {T, F}.

- Γ = {p(x), p(f(x)), q(c, y), q(f(f(x)), x)}

- Consider the previous interpretation in which
  Δ = {0, 1, 2}, μ(c) = 0,
  μ(f) = {0 → 2, 1 → 0, 2 → 1},
  μ(p) = {(2)}
  μ(q) = {(0, 0), (0, 1), (0, 2)}

- What are the values of
  (∀y)q(x, y)
  for the assignment {x → 0}?
- The assignments agreeing with {x → 0} on free((∀y) q(x, y)) = {x} are: {x → 0, y → 0}, {x → 0, y → 1} , {x → 0, y → 2}.

An assignment α for interpretation (Δ, μ) maps the set of **all formulas** to {T, F}.

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- What are the values of
  (∀x)q(x, y)
  for the assignment {y → 0}?
- The assignments agreeing with {y → 0} on free((∀y) q(x, y)) = {x} are: {x → 0, y → 0}, {x → 1, y → 0} , {x → 2, y → 0}.

An assignment α for interpretation (Δ, μ) maps the set of **all formulas** to {T, F}.

- $\Gamma = \{p(x), p(f(x)), q(c, y), q(f(f(x)), x)\}$
- Consider the previous interpretation in which
  $\Delta = \{0, 1, 2\}$, $\mu(c) = 0$,
  $\mu(f) = \{0 \to 2, 1 \to 0, 2 \to 1\}$,
  $\mu(p) = \{(2)\}$
  $\mu(q) = \{(0, 0), (0, 1), (0, 2)\}$
- What are the values of
  $(\forall x)(\forall y)q(x, y)$
  for any assignment, including {}.
- The interpretations agreeing with {} on free $((\forall y)q(x, y)) = \{x\}$ are: $\{x \to 0\}$, $\{x \to 1\}$, $\{x \to 2\}$.

Induced value for formulas $(\exists x)\psi$

- Suppose the formula in question is of the form $(\forall x)\psi$.
- We know that for any assignment α we have a truth value $\alpha(\psi) \in \{T, F\}$.

- Furthermore, those assignments also qualify as assignments for $(\forall x)\psi$.

- We define $\alpha((\exists x)\psi)$ to be T provided that **some** assignment α' for ψ that **agrees with** α on free($(\exists x)\psi$) is such that $\alpha'(\psi) = T$. Otherwise the value $\alpha((\exists x)\psi)$ is F.

An assignment α for interpretation (Δ, μ) maps the set of **all formulas** to {T, F}.

- $\Gamma = \{p(x), p(f(x)), q(c, y), q(f(f(x)), x)\}$
- Consider the previous interpretation in which
  $\Delta = \{0, 1, 2\}$, $\mu(c) = 0$,
  $\mu(f) = \{0 \rightarrow 2, 1 \rightarrow 0, 2 \rightarrow 1\}$,
  $\mu(p) = \{(2)\}$
  $\mu(q) = \{(0, 0), (0, 1), (0, 2)\}$
- What are the values of
  $(\exists y)q(x, y)$
  for the assignment $\{x \rightarrow 0\}$?
- The assignments agreeing with $\{x \rightarrow 0\}$ on free($(\exists y)q(x, y)$) = $\{x\}$ are: $\{x \rightarrow 0, y \rightarrow 0\}$, $\{x \rightarrow 0, y \rightarrow 1\}$, $\{x \rightarrow 0, y \rightarrow 2\}$.

An assignment α for interpretation (Δ, μ) maps the set of **all formulas** to {T, F}.

- Γ = {p(x), p(f(x)), q(c, y), q(f(f(x)), x)}

- Consider the previous interpretation in which
  Δ = {0, 1, 2}, μ(c) = 0,
  μ(f) = {0 → 2, 1 → 0, 2 → 1},
  μ(p) = {(2)}
  μ(q) = {(0, 0), (0, 1), (1, 2)}

- What are the values of
  (∃x)q(x, y)
  for the assignment {y → 0}?
- The interpretations agreeing with {y → 0} on free((∃x)q(x, y)) = {y} are: {x → 0, y → 0}, {x → 1, y → 0} , {x → 2, y → 0}.

An assignment α for interpretation (Δ, μ) maps the set of **all formulas** to {T, F}.

- Γ = {p(x), p(f(x)), q(c, y), q(f(f(x)), x)}

- Consider the previous interpretation in which
  Δ = {0, 1, 2}, μ(c) = 0,
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  μ(q) = {(0, 0), (0, 1), (1, 2)}

- What are the values of
  (∃x)(∃y)q(x, y)
  for any assignment, including {}.
- The assignments agreeing with {} on free ((∃x)(∃y)q(x, y)) = {} are: {x → 0}, {x → 1} , {x → 2}.

Formalizing Entailment |=

- When $\varphi_1, \ldots, \varphi_n, \psi$ are predicate calculus formulas,

  $$\varphi_1, \ldots, \varphi_n \models \psi$$

  means:

  For every interpretation (Δ, μ) for the formulas $\{\varphi_1, \ldots, \varphi_n, \psi\}$

  for every assignment α such that for each i $\alpha(\varphi_i) = T$,

  it must also be the case that $\alpha(\psi) = T$.

Validity

- When the left-hand side is empty: $\models \psi$ we say that is **universally valid**.

|= in predicate calculus vs. propositional

- The predicate version of $\models \psi$ is a **very broad** statement:
  - The set of applicable structures is generally infinite.
  - If a given domain is infinite, so is the set of assignments.
- Intuitively there is much less likely to be an algorithm to check whether $\models \psi$ for predicate calculus in the way there is for the propositional calculus.

Soundness and Completeness

- As with propositional logic, we define:

- **Soundness** of a set of derivation rules:

  For any set of formulas Γ and any formula ψ:
  $\Gamma \vdash \psi$ implies $\Gamma \models \psi$

- **Completeness** of a set of derivation rules:

  For any set of formulas Γ and any formula ψ:
  $\Gamma \models \psi$ implies $\Gamma \vdash \psi$

- **Completeness Theorem**:
  Our natural deduction rules for predicates are both sound and complete (For proof, *cf*. Van Dalen book).

Models

- An **interpretation** for a set of formulas Γ such that

  for every assignment α, $\alpha(\varphi)$ = T for each $\varphi \in \Gamma$

  is called a **model** for Γ.

- $\Gamma \models \psi$ can be restated as:

  Every model for Γ is also a model for $\{\psi\}$.

Validity and Satisfiability

- If $\alpha(\psi) = \top$ for **some** interpretation and some assignment α, we say that ψ is **satisfiable**.

- A set of closed (i.e. no free variables) formulas Γ is satisfiable iff Γ has a model.

- For a closed formula ψ, the following are equivalent:

  - $\Gamma \models \psi$
  - $\Gamma \cup \{\neg\psi\}$ is **un**satisfiable
  - $\Gamma \cup \{\neg\psi\} \models \bot$

- This is the basis for many automatic theorem provers:
  To show $\Gamma \models \psi$, show $\Gamma \cup \{\neg\psi\}$ is **un**satisfiable by deriving $\bot$ from the formulas $\Gamma \cup \{\neg\psi\}$.

Deciding whether or not $\Gamma \mid\!- \psi$

- If we think $\Gamma \mid\!- \psi$, try to find a proof of it.

- If we think not $\Gamma \mid\!- \psi$, try to find a **counterexample** (or "counter model"), i.e. a model for Γ where ψ is false.

- We will later see that there is **no algorithm** to decide which one is the case.

Gödel's Incompleteness Theorem
(preview)

- No consistent […] extension of number theory is complete (in the sense that not all formulas that are true *for this intended interpretation* are derivable).

- In other words, we can ***try*** to build up an **all-powerful mathematical theory**. At a minimum, it must include number theory, which is not asking very much. But such a theory will always be **incomplete**.

- This result, published in 1931, meant that Hilbert's idea of mechanizing all of mathematics could **never** be achieved.

- Kurt Gödel, *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, I.* Monatshefte für Mathematik und Physik, 38 (1931), pp. 173-198. Translated in van Heijenoort: *From Frege to Gödel*. Harvard University Press, 1971., online at http://home.ddc.net/ygg/etext/godel/

Addendum to Semantics of Predicate Calculus: Predicate Calculus with Equality

- There is one exception to the "all interpretations" definitions of validity when the = predicate symbol is being used:
  - **Equality is always interpreted as identity.**
  - Without this, the equality axioms would not be meaningful.