Predicate Calculus
Semantics

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Syntax vs. Semantics

- Predicate logic proofs, in a system such as natural deduction, focus on **syntax**: each formula in the derivation is **mechanically-checkable** to be derivable from earlier formulas using only the given rules.

- The **semantics** or **meaning** of a formula is determined by separate considerations. Each formula is making a statement about some kind of **underlying mathematical structure**.
Interpretations of Formulas

• The structure(s) of interest in specific derivations are generally **not totally specified** in the system of derivation itself.

• Instead, we rely on certain formulas ("axioms") to **characterize** the properties of these structures that are of interest. In natural deduction, these formulas will appear on the left-hand side of a sequent.

• It can then be proved separately that the syntactic rules are in agreement with the semantics of the intended **interpretation**.
What is an “Interpretation”? 

• An **interpretation** for a set of formulas consists of:

  • A **domain**: that contains all individuals of interest.

  • A mapping from **constant symbols** in the formulas to **specific domain elements** (“constants”).

  • A mapping from the **function symbols** in the language to **functions** of corresponding arity mapping n-tuples of domain elements into the.

  • A mapping from the **predicate symbols** in the language to functions of corresponding arity mapping n-tuples of domain elements into \{T, F\}. 
Interpretation \((\Delta, \mu)\)

- An **interpretation** for a set of formulas consists of:
  - A **domain** \(\Delta\): that contains all individuals of interest.
  - For each **constant symbol** \(c\), an element \(\mu(c) \in \Delta\).
  - For each **function symbol** \(f\), a function \(\mu(f): \Delta^n \rightarrow \Delta\).
  - For each **predicate symbol** \(p\), a function \(\mu(p): \Delta^n \rightarrow \{T, F\}\).
Assignments for the Predicate Calculus

- Suppose $(\Delta, \mu)$ is an interpretation for a set $\Gamma$ of formulas.

- An assignment for the interpretation is mapping from the collective free variables in $\Gamma$ to $\Delta$:

  $$\alpha: \text{free}(\Gamma) \rightarrow \Delta$$
Example

- \( \Gamma = \{ p(x), p(f(x)), q(c, y), q(f(f(x)), x) \} \)
- \( \Delta = \{0, 1, 2\} \)
- \( \mu(c) = 0 \)
- \( \mu(f) = \{0 \rightarrow 2, 1 \rightarrow 0, 2 \rightarrow 1\} \)
- \( \mu(p) = \{(2)\} \) [the set of 1-tuples for which \( \mu(p) \) is T]
- \( \mu(q) = \{(0, 0), (0, 1), (0, 2)\} \) [the set of 2-tuples for which \( \mu(q) \) is T]
- Some assignments \( \alpha \) are:
  - \( \{ x \rightarrow 0, y \rightarrow 0 \} \)
  - \( \{ x \rightarrow 0, y \rightarrow 1 \} \)
  - \( \{ x \rightarrow 1, y \rightarrow 2 \} \), etc. [How many altogether?]
An assignment $\alpha$ for interpretation $(\Delta, \mu)$ maps the set of all terms to a value in $\Delta$.

- Let $\alpha$ be an assignment.

- If $t$ is a variable symbol $\nu$, then $\alpha(\nu)$ is just the value to which $\nu$ maps by the interpretation.

- If $t$ is a constant symbol $c$, then $\alpha(c) = \mu(c)$, the value assigned by the interpretation.

- If $t$ is $f(t_1, \ldots, t_n)$ then $\alpha(t) = \mu(f)(\alpha(t_1), \ldots, \alpha(t_n))$. 
An assignment $\alpha$ for interpretation $(\Delta, \mu)$ maps the set of all terms to a value in $\Delta$.

- $\Gamma = \{p(x), p(f(x)), q(c, y), q(f(f(x)), x)\}$

- The terms of interest here are: $x, f(x), c, y, f(f(x))$

- Consider the previous interpretation in which $\Delta = \{0, 1, 2\}$, $\mu(c) = 0$, $\mu(f) = \{0 \rightarrow 2, 1 \rightarrow 0, 2 \rightarrow 1\}$, what are the induced values for assignment $\{x \rightarrow 0, y \rightarrow 0\}$?
An assignment \( \alpha \) for interpretation \((\Delta, \mu)\) maps the set of all terms to a value in \( \Delta \).

- \( \Gamma = \{p(x), p(f(x)), q(c, y), q(f(f(x)), x)\} \)

- The terms of interest here are:
  \( x, f(x), c, y, f(f(x)) \)

- Consider the previous interpretation in which \( \Delta = \{0, 1, 2\}, \mu(c) = 0, \)
  \( \mu(f) = \{0 \rightarrow 2, 1 \rightarrow 0, 2 \rightarrow 1\} \), what are the induced values for assignment

  \( \{x \rightarrow 1, y \rightarrow 2\}? \)
An assignment $\alpha$ for interpretation $(\Delta, \mu)$ maps every atomic formula to $\{T, F\}$.

- An atomic formula has the form $p(t_1, ..., t_n)$ where $p$ is a predicate symbol and are terms.

- We already defined $\alpha(t_i)$ for terms $t_i$.

- $\alpha(p(t_1, ..., t_n))$ is defined as $\mu(p)(\alpha(t_1), ..., \alpha(t_n))$.

- Note: An atomic formula might have no free variables. Then the value of any assignment is determined entirely by the interpretation itself.

- Note: An assignment mapping the empty set of variable symbols has the value constant $T$ or constant $F$, as determined by the interpretation.
An assignment $\alpha$ for interpretation $(\Delta, \mu)$ maps the set of *atomic* formulas to $\{T, F\}$.

- $\Gamma = \{ p(x), p(f(x)), q(c, y), q(f(f(x)), x) \}$

- Consider the previous interpretation in which
  \[ \Delta = \{0, 1, 2\}, \mu(c) = 0, \]
  \[ \mu(f) = \{0 \rightarrow 2, 1 \rightarrow 0, 2 \rightarrow 1\}, \]
  \[ \mu(p) = \{(2)\} \]
  \[ \mu(q) = \{(0, 0), (0, 1), (0, 2)\} \]

- What are the values for the atomic formulas for the assignment
  \[ \{x \rightarrow 0, y \rightarrow 0\}? \]
An assignment $\alpha$ for interpretation $(\Delta, \mu)$ maps the set of *atomic* formulas to \{T, F\}.

- $\Gamma = \{p(x), p(f(x)), q(c, y), q(f(f(x)), x)\}$

- Consider the previous interpretation in which
  $\Delta = \{0, 1, 2\}$, $\mu(c) = 0$,
  $\mu(f) = \{0 \rightarrow 2, 1 \rightarrow 0, 2 \rightarrow 1\}$,
  $\mu(p) = \{(2)\}$
  $\mu(q) = \{(0, 0), (0, 1), (0, 2)\}$

- What are the values for the atomic formulas for the assignment
  $\{x \rightarrow 1, y \rightarrow 2\}$?
An assignment $\alpha$ for interpretation $(\Delta, \mu)$ maps the set of quantifier-free formulas to \{T, F\}.

- Quantifier-free formulas are just those built from atomic formulas using the propositional connectives $\neg \land \lor \rightarrow$.

- Since we know the induced values for atomic formulas, we get the induced values for any quantifier-free formula using the same definitions as for propositional calculus.
Assignments

- For any formula of the form \((\varphi \lor \psi)\):
  \[
  \alpha(\varphi \lor \psi) = T \text{ iff } \alpha(\varphi) = T \text{ or } \alpha(\psi) = T.
  \]

- For any formula of the form \((\varphi \land \psi)\):
  \[
  \alpha(\varphi \land \psi) = T \text{ iff } \alpha(\varphi) = T \text{ and } \alpha(\psi) = T.
  \]

- For any formula of the form \((\varphi \rightarrow \psi)\):
  \[
  \alpha(\varphi \rightarrow \psi) = T \text{ iff } \alpha(\varphi) = F \text{ or } \alpha(\psi) = T.
  \]

- For any formula of the form \((\neg \varphi)\):
  \[
  \alpha(\neg \varphi) = T \text{ iff } \alpha(\varphi) = F.
  \]
An assignment $\alpha$ for interpretation $(\Delta, \mu)$ maps the set of all formulas to $\{T, F\}$.

- Since we know the induced values for quantifier-free formulas, we need to show how the values for formulas in general are defined.
- In short, we need to show what happens when the formula contains quantifiers.
- Then we again use propositional rules to determine the value for arbitrary formulas (since these can be formed by combining non-quantifier-free formulas).
Induced value for formulas $\forall x \psi$

- Suppose the formula in question is of the form $\forall x \psi$.
- We know that for any assignment $\alpha$ we have a truth value $\alpha(\psi) \in \{T, F\}$.
- Furthermore, those assignments also qualify as assignments for $\forall x \psi$.
- We define $\alpha((\forall x)\psi)$ to be $T$ provided that every assignment $\alpha'$ for $\psi$ that agrees with $\alpha$ on free($\forall x \psi$) is such that $\alpha'(\psi) = T$. Otherwise the value $\alpha((\forall x)\psi)$ is $F$. 
An assignment \( \alpha \) for interpretation \((\Delta, \mu)\) maps the set of all formulas to \{T, F\}.

- \( \Gamma = \{p(x), p(f(x)), q(c, y), q(f(f(x)), x)\} \)

- Consider the previous interpretation in which
  \( \Delta = \{0, 1, 2\} \), \( \mu(c) = 0 \),
  \( \mu(f) = \{0 \rightarrow 2, 1 \rightarrow 0, 2 \rightarrow 1\} \),
  \( \mu(p) = \{(2)\} \)
  \( \mu(q) = \{(0, 0), (0, 1), (0, 2)\} \)

- What are the values of \((\forall y)q(x, y)\) for the assignment \( \{x \rightarrow 0\} \)?

- The assignments agreeing with \( \{x \rightarrow 0\} \) on \( \text{free}(\forall y \ q(x, y)) = \{x\} \) are: \( \{x \rightarrow 0, y \rightarrow 0\} \), \( \{x \rightarrow 0, y \rightarrow 1\} \), \( \{x \rightarrow 0, y \rightarrow 2\} \).
An assignment $\alpha$ for interpretation $(\Delta, \mu)$ maps the set of **all formulas** to $\{T, F\}$.

- $\Gamma = \{p(x), p(f(x)), q(c, y), q(f(f(x)), x)\}$

- Consider the previous interpretation in which
  $\Delta = \{0, 1, 2\}$, $\mu(c) = 0$,
  $\mu(f) = \{0 \rightarrow 2, 1 \rightarrow 0, 2 \rightarrow 1\}$,
  $\mu(p) = \{(2)\}$
  $\mu(q) = \{(0, 0), (0, 1), (0, 2)\}$

- What are the values of $(\forall x)q(x, y)$ for the assignment $\{y \rightarrow 0\}$?

- The assignments agreeing with $\{y \rightarrow 0\}$ on $\text{free}((\forall y) q(x, y)) = \{x\}$ are: $\{x \rightarrow 0, y \rightarrow 0\}$, $\{x \rightarrow 1, y \rightarrow 0\}$, $\{x \rightarrow 2, y \rightarrow 0\}$. 
An assignment $\alpha$ for interpretation $(\Delta, \mu)$ maps the set of all formulas to $\{T, F\}$.

- $\Gamma = \{p(x), p(f(x)), q(c, y), q(f(f(x)), x)\}$

- Consider the previous interpretation in which
  $\Delta = \{0, 1, 2\}$, $\mu(c) = 0$,
  $\mu(f) = \{0 \rightarrow 2, 1 \rightarrow 0, 2 \rightarrow 1\}$,
  $\mu(p) = \{(2)\}$
  $\mu(q) = \{(0, 0), (0, 1), (0, 2)\}$

- What are the values of $(\forall x)(\forall y)q(x, y)$ for any assignment, including $\{\}$.
- The interpretations agreeing with $\{\}$ on free $((\forall y)q(x, y)) = \{x\}$ are: $\{x \rightarrow 0\}$, $\{x \rightarrow 1\}$, $\{x \rightarrow 2\}$. 
Induced value for formulas \((\exists x)\psi\)

- Suppose the formula in question is of the form \((\forall x)\psi\).
- We know that for any assignment \(\alpha\) we have a truth value \(\alpha(\psi) \in \{T, F\}\).
- Furthermore, those assignments also qualify as assignments for \((\forall x)\psi\).
- We define \(\alpha((\exists x)\psi)\) to be \(T\) provided that some assignment \(\alpha'\) for \(\psi\) that agrees with \(\alpha\) on free((\exists x)\psi)\) is such that \(\alpha'(\psi) = T\). Otherwise the value \(\alpha((\exists x)\psi)\) is \(F\).
An assignment $\alpha$ for interpretation $(\Delta, \mu)$ maps the set of all formulas to $\{T, F\}$.

- $\Gamma = \{p(x), p(f(x)), q(c, y), q(f(f(x)), x)\}$

- Consider the previous interpretation in which
  $\Delta = \{0, 1, 2\}$, $\mu(c) = 0$,
  $\mu(f) = \{0 \rightarrow 2, 1 \rightarrow 0, 2 \rightarrow 1\}$,
  $\mu(p) = \{(2)\}$
  $\mu(q) = \{(0, 0), (0, 1), (0, 2)\}$

- What are the values of $(\exists y)q(x, y)$ for the assignment $\{x \rightarrow 0\}$?
- The assignments agreeing with $\{x \rightarrow 0\}$ on $\text{free}((\exists y)q(x, y)) = \{x\}$ are: $\{x \rightarrow 0, y \rightarrow 0\}$, $\{x \rightarrow 0, y \rightarrow 1\}$, $\{x \rightarrow 0, y \rightarrow 2\}$. 
An assignment $\alpha$ for interpretation $(\Delta, \mu)$ maps the set of **all formulas** to $\{T, F\}$.

- $\Gamma = \{p(x), p(f(x)), q(c, y), q(f(f(x)), x)\}$

- Consider the previous interpretation in which
  $\Delta = \{0, 1, 2\}$, $\mu(c) = 0$,
  $\mu(f) = \{0 \rightarrow 2, 1 \rightarrow 0, 2 \rightarrow 1\}$,
  $\mu(p) = \{(2)\}$
  $\mu(q) = \{(0, 0), (0, 1), (1, 2)\}$

- What are the values of $(\exists x)q(x, y)$ for the assignment $\{y \rightarrow 0\}$?
- The interpretations agreeing with $\{y \rightarrow 0\}$ on $\text{free}((\exists x)q(x, y)) = \{y\}$ are: $\{x \rightarrow 0, y \rightarrow 0\}$, $\{x \rightarrow 1, y \rightarrow 0\}$, $\{x \rightarrow 2, y \rightarrow 0\}$. 
An assignment $\alpha$ for interpretation $(\Delta, \mu)$ maps the set of all formulas to $\{T, F\}$.

- $\Gamma = \{p(x), p(f(x)), q(c, y), q(f(f(x)), x)\}$

- Consider the previous interpretation in which
  $\Delta = \{0, 1, 2\}$, $\mu(c) = 0$,
  $\mu(f) = \{0 \rightarrow 2, 1 \rightarrow 0, 2 \rightarrow 1\}$,
  $\mu(p) = \{(2)\}$
  $\mu(q) = \{(0, 0), (0, 1), (1, 2)\}$

- What are the values of $(\exists x)(\exists y)q(x, y)$ for any assignment, including $\{}$.
- The assignments agreeing with $\{}$ on free $((\exists x)(\exists y)q(x, y)) = \{}$ are: $\{x \rightarrow 0\}$, $\{x \rightarrow 1\}$, $\{x \rightarrow 2\}$. 
Formalizing Entailment $|=\$

- When $\varphi_1, \ldots, \varphi_n, \psi$ are predicate calculus formulas,

\[ \varphi_1, \ldots, \varphi_n |= \psi \]

means:

For every interpretation $(\Delta, \mu)$ for the formulas $\{\varphi_1, \ldots, \varphi_n, \psi\}$

for every assignment $\alpha$ such that for each $i$ $\alpha(\varphi_i) = T$, 

it must also be the case that $\alpha(\psi) = T$. 
Validity

- When the left-hand side is empty: $\models \psi$ we say that is **universally valid**.
|= in predicate calculus vs. propositional

- The predicate version of |= ψ is a very broad statement:
  - The set of applicable structures is generally infinite.
  - If a given domain is infinite, so is the set of assignments.

- Intuitively there is much less likely to be an algorithm to check whether |= ψ for predicate calculus in the way there is for the propositional calculus.
Soundness and Completeness

- As with propositional logic, we define:

- **Soundness** of a set of derivation rules:
  
  For any set of formulas $\Gamma$ and any formula $\psi$:
  
  $\Gamma \vdash \psi$ implies $\Gamma \models \psi$

- **Completeness** of a set of derivation rules:
  
  For any set of formulas $\Gamma$ and any formula $\psi$:
  
  $\Gamma \models \psi$ implies $\Gamma \vdash \psi$

- **Completeness Theorem**:
  
  Our natural deduction rules for predicates are both sound and complete (For proof, cf. Van Dalen book).
Models

• An **interpretation** for a set of formulas $\Gamma$ such that for every assignment $\alpha$, $\alpha(\varphi) = T$ for each $\varphi \in \Gamma$ is called a **model** for $\Gamma$.

• $\Gamma \models \psi$ can be restated as:

  Every model for $\Gamma$ is also a model for $\{\psi\}$.
Validity and Satisfiability

- If $\alpha(\psi) = T$ for some interpretation and some assignment $\alpha$, we say that $\psi$ is **satisfiable**.

- A set of closed (i.e. no free variables) formulas $\Gamma$ is satisfiable iff $\Gamma$ has a model.

- For a closed formula $\psi$, the following are equivalent:
  
  - $\Gamma \models \psi$
  - $\Gamma \cup \{\neg \psi\}$ is **unsatisfiable**
  - $\Gamma \cup \{\neg \psi\} \models \bot$

- This is the basis for many automatic theorem provers:
  To show $\Gamma \models \psi$, show $\Gamma \cup \{\neg \psi\}$ is **unsatisfiable** by deriving $\bot$ from the formulas $\Gamma \cup \{\neg \psi\}$. 
Deciding whether or not $\Gamma \vdash \psi$

- If we think $\Gamma \vdash \psi$, try to find a proof of it.

- If we think not $\Gamma \vdash \psi$, try to find a counterexample (or “counter model”), i.e. a model for $\Gamma$ where $\psi$ is false.

- We will later see that there is no algorithm to decide which one is the case.
Gödel’s Incompleteness Theorem
(preview)

• No consistent [...] extension of number theory is complete (in the sense that not all formulas that are true for this intended interpretation are derivable).

• In other words, we can try to build up an all-powerful mathematical theory. At a minimum, it must include number theory, which is not asking very much. But such a theory will always be incomplete.

• This result, published in 1931, meant that Hilbert’s idea of mechanizing all of mathematics could never be achieved.

Addendum to Semantics of Predicate Calculus: Predicate Calculus with Equality

• There is one exception to the "all interpretations" definitions of validity when the = predicate symbol is being used:
  
  • **Equality is always interpreted as identity.**
  
  • Without this, the equality axioms would not be meaningful.