Propositional Semantics

Robert Keller

26 January 2005

Syntax vs. Semantics

- In a logical system, proofs are constructed using principles of **syntax**:
  - Symbol manipulation is used.
  - There is no reliance on specific meaning of the formulas. Each step can be mechanically checked (i.e. there is an algorithm for checking each step)
  - Entire proofs can be mechanically checked

- In contrast, we would like a companion notion of **semantics** or meaning of each expression.

Semantics of Propositional Logic

- The standard semantics for propositional logic is in terms of a 2-valued domain of truth values, using **truth functions**.

- This is not the only possible semantics. For example, there are various n-valued logics ($n > 2$), fuzzy logic, etc. which could also be applied.

Semantics of Propositional Formulas

- An **assignment** is a function that assigns a truth value in {T, F} to each proposition symbol (a, b, c, …) in a set of formulas.

- For example, consider the set of two formulas $\{(p \wedge q) \rightarrow r,\ \ p \rightarrow (q \rightarrow r)\}$.
  - One assignment is $\alpha(p)=F$, $\alpha(q)= F$, $\alpha(r)= F$.
  - Another assignment is $\alpha(p)=F$, $\alpha(q)= F$, $\alpha(r)= T$.

- For n distinct proposition symbols, how many distinct assignments are there?

The Induced Value of a Formula

- An **assignment** α for the propositions in a set of formulas **induces** a value $\alpha(\varphi)$ for each formula φ according to the following recursive rules:
  - If φ is a proposition variable ν by itself, then $\alpha(\varphi) = \alpha(\nu)$.
  - $\alpha(\perp) = F$
  - $\alpha(T) = T$
  - $\alpha(\neg\varphi)$ = T if $\alpha(\varphi) = F$; F otherwise.
  - $\alpha(\varphi \wedge \psi)$ = T if $\alpha(\varphi) = T$ **and** $\alpha(\psi) = T$; F otherwise.
  - $\alpha(\varphi \vee \psi)$ = T if $\alpha(\varphi) = T$ **or** $\alpha(\psi) = T$; F otherwise.
  - $\alpha(\varphi \rightarrow \psi)$ = T if $\alpha(\varphi) = F$ or $\alpha(\psi) = T$; F otherwise.

Example

- Compute the induced values of the formulas in $\{(p \wedge q) \rightarrow r, \ p \rightarrow (q \rightarrow r)\}$ under the assignment $\alpha(p)=T$, $\alpha(q)= F$, $\alpha(r)= T$.

Semantic Entailment

- $\varphi_1, \ldots, \varphi_n \models \psi$ means the following:

- For each assignment α such that

$$\alpha(\varphi_1) = \alpha(\varphi_2) = \ldots = \alpha(\varphi_n) = \text{true},$$

  it is also the case that $\alpha(\psi) = \text{true}$.

- When the LHS is **empty**, $\models \psi$, we say that ψ is a **tautology**.

Example of $\models$

- Check whether $p \rightarrow (q \vee r) \models (p \rightarrow q) \vee (p \rightarrow r)$
- Check whether $p, (q \rightarrow p) \models q$

Method for checking $\models \psi$

- There are numerous methods actually.
- One method I find easy to remember is the Boole/Shannon method (from CS 60), called "Quine's method" in Hein.
- If ψ has no variables, then for any assignment, it induces either T or F independent of the assignment, and only in the latter case does $\models \psi$ hold.
- Otherwise, pick any variable ν in ψ and check whether both $\models \psi[T/\nu]$ and $\models \psi[\perp/\nu]$, where

  $\psi[\eta/\nu]$ designates the result of replacing ν with η throughout ψ.

Extending the Check to $\varphi_1, \ldots, \varphi_n \models \psi$

- We can establish the following useful meta-fact:

  $$\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$$

  iff

  $$\models \varphi_1 \rightarrow (\varphi_2 \rightarrow (\ldots \rightarrow (\varphi_n \rightarrow \psi)) \ldots)$$

  This can be proved by mathematical induction on n.

Proof by Induction on n

- Basis:
  For n = 0, the statement becomes
  $\models \psi$ iff $\models \psi$, which is obviously true.

Proof by Induction on n, continued

- Assume the induction hypothesis:
  for all $\varphi_1, \varphi_2, \ldots, \varphi_n, \psi$
  $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$ iff $\models \varphi_1 \rightarrow (\varphi_2 \rightarrow (\ldots \rightarrow (\varphi_n \rightarrow \psi)) \ldots)$,
  to show
  $\varphi_1, \varphi_2, \ldots, \varphi_n, \varphi_{n+1} \models \psi$ iff $\models \varphi_1 \rightarrow (\varphi_2 \rightarrow (\ldots \rightarrow (\varphi_{n+1} \rightarrow \psi))\ldots)$.

($\Rightarrow$) Assume $\varphi_1, \varphi_2, \ldots, \varphi_n, \varphi_{n+1} \models \psi$.

Suppose α is an assignment s.t. $\alpha(\varphi_1) = \ldots = \alpha(\varphi_{n+1}) = T$.
Then by definition of $\models$, $\alpha(\psi) = T$ also.
Therefore $\alpha(\varphi_{n+1} \rightarrow \psi) = T$ by the definition of α for $\rightarrow$.
Thus $\varphi_1, \varphi_2, \ldots, \varphi_n \models (\varphi_{n+1} \rightarrow \psi)$.
So by the induction hypothesis,
$\models \varphi_1 \rightarrow (\varphi_2 \rightarrow (\ldots \rightarrow (\varphi_n \rightarrow (\varphi_{n+1} \rightarrow \psi)) \ldots)$ where $(\varphi_{n+1} \rightarrow \psi)$ is playing the role of ψ in the induction hypothesis.

Proof by Induction on n, continued

($\Leftarrow$) Assume $\models \varphi_1 \rightarrow (\varphi_2 \rightarrow (\ldots \rightarrow (\varphi_{n+1} \rightarrow \psi))\ldots)$.

Then identifying $(\varphi_{n+1} \rightarrow \psi)$ with ψ in the induction hypothesis, we have $\varphi_1, \varphi_2, \ldots, \varphi_n \models (\varphi_{n+1} \rightarrow \psi)$.

Let be an assignment α such that $\alpha(\varphi_1) = \ldots = \alpha(\varphi_{n+1}) = T$.

In particular, $\alpha(\varphi_1) = \ldots = \alpha(\varphi_n) = T$.

So by definition of $\models$, $\alpha(\varphi_{n+1} \rightarrow \psi) = T$ also.

But since $\alpha(\varphi_{n+1}) = T$, we must have $\alpha(\psi) = T$ as well by definition of α for $\rightarrow$.

Therefore

$\varphi_1, \varphi_2, \ldots, \varphi_n, \varphi_{n+1} \models \psi$ iff $\models \varphi_1 \rightarrow (\varphi_2 \rightarrow (\ldots \rightarrow (\varphi_{n+1} \rightarrow \psi))\ldots)$.

Summary of Entailment vs. Provability

- $\varphi_1, \ldots, \varphi_n \vdash \psi$ means ψ is **provable** from $\varphi_1, \ldots, \varphi_n$
- $\varphi_1, \ldots, \varphi_n \models \psi$ means (the truth of) $\varphi_1, \ldots, \varphi_n$

  "entails" (the truth of) ψ
- It would be nice if these two separately defined concepts coincided. Why?

Advantages of $\vdash$ coinciding with $\models$

- There is an **algorithm** for checking $\models$.

  Therefore we could test whether a sequent is true before setting out to prove it.

  If it is not true, there is no point in trying.

- It would be reassuring if we knew that every true statement were provable.

Reasoning with |=

- We use this to show soundness, for example:

  $$\varphi_1, \ldots, \varphi_n, \varphi \models \psi$$

  iff

  $$\varphi_1, \ldots, \varphi_n \models (\varphi \to \psi)$$

- Proof ($\Rightarrow$) Suppose $\varphi_1, \ldots, \varphi_n, \varphi \models \psi$. Let α be an assignment such that $\alpha(\varphi_1) = \ldots = \alpha(\varphi_n) = T$.

- If also $\alpha(\varphi) = T$, then $\alpha(\psi) = T$ by our supposition.

- So $\alpha(\varphi \to \psi) = T$. If $\alpha(\varphi) = F$, then $\alpha(\varphi \to \psi) = T$ by definition of $\to$.

- Therefore $\varphi_1, \ldots, \varphi_n \models (\varphi \to \psi)$.

Reasoning with |=

- Proof ($\Leftarrow$) Suppose $\varphi_1, \ldots, \varphi_n \models (\varphi \rightarrow \psi)$
- Let α be an assignment such that $\alpha(\varphi_1) = \ldots = \alpha(\varphi_n) = T$.
- Then $\alpha(\varphi \rightarrow \psi) = T$.
- If also $\alpha(\varphi) = T$, then $\alpha(\psi) = T$ by definition of $\rightarrow$.
- Therefore $\varphi_1, \ldots, \varphi_n, \varphi \models \psi$.

Soundness

- A logical system $\vdash$ is **sound** if

  $$\varphi_1, \ldots, \varphi_n \vdash \psi \text{ implies } \varphi_1, \ldots, \varphi_n \models \psi$$

  for all formulas $\varphi_1, \ldots, \varphi_n, \psi$.

  In a sound system, every provable sequent is semantically correct.

- Another word for "sound" is "**consistent**".

How could a system not be sound?

- Easy: If we are over-zealous in our choice of rules, we can add rules that permit absurd conclusions.
- Example: deriving $\perp$ from an empty set of premises, which would allow the derivation of any formula whatsoever (using $\perp$ elimination).

Completeness

- A logical system $\vdash$ is **complete** if

  $$\varphi_1, \ldots, \varphi_n \models \psi \text{ implies } \varphi_1, \ldots, \varphi_n \vdash \psi$$

  for all formulas $\varphi_1, \ldots, \varphi_n, \psi$.

  In a complete system, every semantically correct sequent is provable.

How could a system not be complete?

- If we aren’t careful, we can fail to include rules that are needed to prove certain true statements.

- Intuitionistic logic is not complete by design. For example, LEM can’t be derived.

Metatheorem

- The natural deduction system presented for the propositional calculus (including the non-intuitionistic rule $\neg\neg$e) is both sound and complete.

How to prove soundness?

- Induction of some form must be used.
- In general, we must show:
  - Each axiom is a tautology.
  - Each rule is “truth-preserving”.
    - If the antecedents are true, then the consequent is true.
- For our particular system, there are no axioms, so that part is simple.
- For the rules without sub-derivations, it is a matter of showing truth preservation for each rule.
- For the rules with sub-deriviations, things are a little trickier.

Proof of Soundness

- We will use the "strong form" of mathematical induction on the
  length of the derivation D of $\varphi_1, \ldots, \varphi_n \vdash \psi$ (this length is not n).
- Let P(k) be:
  For any derivation D of length k of
  $\varphi_1, \ldots, \varphi_n \vdash \psi$ it is the case that $\varphi_1, \ldots, \varphi_n \models \psi$.

- The strong induction principle is:

  For all k $\geq$ 0,
  (for all k' < k P(k')) implies P(k),

  then for all k $\geq$ 0 P(k).

Proof by Strong Induction

- The case $P(0)$ must be proved directly, since (for all $k' < 0$ $P(k')$) is vacuously true
  (there is no proof of length < 0).

- A proof of length 0 can only be that ψ is one of the premises $\varphi_1, \ldots, \varphi_n$.
  Therefore $\varphi_1, \ldots, \varphi_n \models \psi$.

Proof by Strong Induction

- We now want to show that for all $k \geq 1$, (for all $k' < k$ $P(k')$) implies $P(k)$.
- Assume a derivation of $\varphi_1, \ldots, \varphi_n \vdash \psi$ having length $k \geq 1$.
- The last line of the derivation is ψ, and it is either a copy of a premise or an earlier line, or was derived using a **rule**.
- The first two cases are covered by the induction hypothesis, so we need only consider the case of derivation by a rule.
- The rule either used formulas that occurred earlier in the derivation, premises, or sub-proofs as antecedent.

Case where the antecedents are formulas

- For the rule to be applicable, the formulas in the antecedents must have occurred earlier in the proof.

- If we can show that each such rule **preserves truth**, then we have completed this case $\varphi_1, \ldots, \varphi_n \models \psi$.

Truth Preservation for Rules without Sub-Derivations

- If $\varphi_1, \ldots, \varphi_n \vdash \psi$ then ψ is derived as the consequent of some **rule**

  $$\frac{\gamma_1 \; . \; . \; . \gamma_k}{\psi}$$

  where each γ_i is in $\varphi_1, \ldots, \varphi_n$.

- Check for each rule that **if** α is an assignment where $\alpha(\gamma_1) = . \; . \; . = \alpha(\gamma_k) = T$, **then** necessarily $\alpha(\psi) = T$.

- But since each γ_i was derived earlier in the proof in order this rule to be applicable, we have $\varphi_1, \ldots, \varphi_n \models \psi$.

Truth Preservation for ($\wedge e_1$, $\wedge e_2$)

- $\dfrac{\varphi \wedge \psi}{\varphi}$ ($\wedge e_1$)

- $\dfrac{\varphi \wedge \psi}{\psi}$ ($\wedge e_2$)

- If $\alpha(\varphi \wedge \psi) = T$, then it *must* be the case that $\alpha(\varphi) = \alpha(\psi) = T$, by definition of the induced value for $\wedge$.

Truth Preservation for $(\vee i_1, \vee i_2)$

- $$\frac{\varphi}{\varphi \vee \psi} \qquad (\vee i_1)$$
- $$\frac{\psi}{\varphi \vee \psi} \qquad (\vee i_2)$$
- In the first case, if $\alpha(\varphi) = T$, then $\alpha(\varphi \vee \psi) = T$.
- In the second case, if $\alpha(\psi) = T$, then $\alpha(\varphi \vee \psi) = T$.

Truth Preservation for $\rightarrow$e

- $$\frac{\varphi, \quad \varphi \rightarrow \psi}{\psi} \quad (\rightarrow e)$$

- If $\alpha(\varphi)$ = T and $\alpha(\varphi \rightarrow \psi)$ = T, then necessarily $\alpha(\psi)$ = T, by definition of the induced α for $\rightarrow$.

Truth Preservation for $\neg\neg$e

- $\dfrac{\neg\neg\varphi}{\varphi}$ ($\neg\neg$e)

- An assignment α such that $\alpha(\neg\neg\varphi) = T$ will have $\alpha(\neg\varphi) = F$, and thus $\alpha(\varphi) = T$.

Truth Preservation for $\neg$e

$$\frac{\varphi \wedge \neg\varphi}{\perp} \quad (\neg e)$$

There is **no** assignment α that makes
$\alpha(\varphi \wedge \neg\varphi) = T$,

so every such assignment makes $\alpha(\perp) = T$.

Truth Preservation for $\perp$e

$$\frac{\perp}{\varphi} \quad (\perp e)$$

There is **no** assignment α that makes $\alpha(\perp) = T$, so every such assignment makes $\alpha(\varphi) = T$.

Truth Preservation for Rules *with* Sub-Derivations: $\rightarrow$i

$$\frac{\boxed{\begin{array}{c}\varphi\\ \cdot\\ \cdot\\ \psi\end{array}}}{\varphi\rightarrow\psi}\ (\rightarrow\text{i})$$

- Consider the $\rightarrow$i rule, which has a sub-derivation φ ... ψ as antecedent and $\varphi \rightarrow \psi$ as consequent. That sub-derivation occurs **within** the proof.
- We can treat the assumption φ **as if** it were one of the premises, and in so doing, derive ψ.
- Since the length of the sub-derivation is less than that of the overall derivation, we can conclude $\varphi_1, \ldots, \varphi_n, \varphi \models \psi$ by the induction hypothesis.
- Then from the definition of $\rightarrow$ and $\models$, we have $\varphi_1, \ldots, \varphi_n \models (\varphi \rightarrow \psi)$.

Truth Preservation for Rules *with* Sub-Derivations: $\vee e$

$$\frac{\varphi \vee \psi \quad \boxed{\begin{matrix}\varphi \\ . \\ . \\ \chi\end{matrix}} \quad \boxed{\begin{matrix}\psi \\ . \\ . \\ \chi\end{matrix}}}{\chi}\ (\vee e)$$

- The $\vee e$ rule has two sub-derivation $\varphi \ldots \chi$ and $\psi \ldots \chi$ in the antecedent. Those sub-derivations occur within the proof.

- We can treat the assumptions as if they were one of the premises, and in so doing, derive χ in both cases.

- Since the length of the sub-derivations is less than that of the overall derivation, we can conclude $\varphi_1, \ldots, \varphi_n, \varphi \models \chi$ and $\varphi_1, \ldots, \varphi_n, \psi \models \chi$ by the induction hypothesis, and thus $\varphi_1, \ldots, \varphi_n, (\varphi \vee \psi) \models \chi$.

- But to use this rule in our proof, we must also have $\varphi_1, \ldots, \varphi_n \models (\varphi \vee \psi)$, so conclude $\varphi_1, \ldots, \varphi_n \models \chi$.

Truth Preservation for Rules *with* Sub-Derivations: ¬i

$$\frac{\boxed{\begin{array}{c}\varphi\\ \vdots\\ \perp\end{array}}}{\neg\varphi}\ (\neg i)$$

- Suppose there is a sub-derivation $\varphi \ldots \perp$ within the proof.
- We can treat the assumption φ as if it were one of the premises, and in so doing, derive $\perp$.
- Since the length of the sub-derivation is less than that of the overall derivation, we can conclude $\varphi_1, \ldots, \varphi_n, \varphi \models \perp$ by the induction hypothesis.
- But $\alpha(\perp) = F$ for all assignments α. Therefore any assignment α such that $\alpha(\varphi_1) = \ldots = \alpha(\varphi_n) = T$ must be such that $\alpha(\varphi) = F$, which is the same as saying $\alpha(\neg\varphi) = T$.
- Thus $\varphi_1, \ldots, \varphi_n \models \neg\varphi$.

Proof of Completeness

Three steps are used:

1. $\varphi_1, \ldots, \varphi_n \models \psi$ implies $\models (\varphi_1 \rightarrow (\varphi_2 \rightarrow . . . (\varphi_n \rightarrow \psi\ ))\ \ldots)$

2. For any formula η, including the r.h.s. above

   $\models \eta$ implies $\vdash \eta$.

3. $\vdash (\varphi_1 \rightarrow (\varphi_2 \rightarrow . . . (\varphi_n \rightarrow \psi\ ))\ \ldots)$ implies $\varphi_1, \ldots, \varphi_n \vdash \psi$

   Step 2 is the key one, as only it bridges the gap between $\models$ and $\vdash$. The other two are merely simplifying steps.

Proofs of 1 and 3

1. $\varphi_1, \ldots, \varphi_n \models \psi$ implies $\models (\varphi_1 \rightarrow (\varphi_2 \rightarrow \ldots (\varphi_n \rightarrow \psi)) \ldots)$
   We proved this earlier, as the first inductive proof in these notes.

3. $\vdash (\varphi_1 \rightarrow (\varphi_2 \rightarrow \ldots (\varphi_n \rightarrow \psi)) \ldots)$ implies $\varphi_1, \ldots, \varphi_n \vdash \psi$
   is proved similarly, using induction and modus ponens.

Proof that $\models \eta$ implies $\vdash \eta$

- Assume $\models \eta$, to show $\models \eta$.

- Let $p_1, p_2, \ldots, p_k$ be the set of all proposition symbols that occur in η.

- For each combination of proposition symbols with and without negation, we will show that there is a provable sequent with that combination on the left and the formula of interest on the right:
  - $p_1, \; p_2, \ldots, p_k \vdash \eta$
  - $\neg p_1, \; p_2, \ldots, p_k \vdash \eta$
  - $p_1, \neg p_2, \ldots, p_k \vdash \eta$
  - $\neg p_1, \neg p_2, \ldots, p_k \vdash \eta$ etc.
- Those 2^k sequents will then be combined into a **single** sequent of the required form.

Example

- Consider the tautology $p \rightarrow (p \vee q)$.
- The sequents to be established are:
  - $\neg p, \neg q \mid\!- p \rightarrow (p \vee q)$, let the derivation be $D_{\neg p, \neg q}$
  - $\neg p, \quad q \mid\!- p \rightarrow (p \vee q)$, let the derivation be $D_{\neg p, q}$
  - $p, \neg q \mid\!- p \rightarrow (p \vee q)$, let the derivation be $D_{p, \neg q}$
  - $p, \quad q \mid\!- p \rightarrow (p \vee q)$, let the derivation be $D_{p, q}$
- Once we have these derivations, we can use nested applications of LEM and $\vee$e to get $\mid\!- p \rightarrow (p \vee q)$.

- (For this particular case, it is obvious how to derive proofs each sequent. We don't even need the premises. However, the method to be described is uniform, and does not rely on obviousness.)

Proof Constructed for the Single Sequent

1.	$p \vee \neg p$	LEM
2.	p	Assumption
3.	$q \vee \neg q$	LEM
4.	q	Assumption
	$\ldots D_{p,\, q} \ldots$	
5.	$p \rightarrow (p \vee q)$	
6.	$\neg q$	Assumption
	$\ldots D_{p,\, \neg q} \ldots$	
7.	$p \rightarrow (p \vee q)$	
8.	$p \rightarrow (p \vee q)$	$\vee$e 3, 4-5, 6-7
9.	$\neg p$	Assumption
10.	$q \vee \neg q$	LEM
11.	q	Assumption
	$\ldots D_{\neg p,\, q} \ldots$	
12.	$p \rightarrow (p \vee q)$	
13.	$\neg q$	Assumption
	$\ldots D_{\neg p,\, \neg q} \ldots$	
14.	$p \rightarrow (p \vee q)$	
15.	$p \rightarrow (p \vee q)$	$\vee$e 10, 11-12, 13-14
16.	$p \rightarrow (p \vee q)$	$\vee$e 1, 2-8, 9-15

Proofs for the Individual Sequents

- For any tautology η, we want to show that $\models \eta$ implies that each of the individual sequents below has a derivation:
  - $p_1, \ p_2, \ldots, p_k \vdash \eta$
  - $\neg p_1, \ p_2, \ldots, p_k \vdash \eta$
  - $p_1, \neg p_2, \ldots, p_k \vdash \eta$
  - $\neg p_1, \neg p_2, \ldots, p_k \vdash \eta$ etc.

where $p_1, p_2, \ldots, p_k$ are the proposition symbols in η.

Generalization

- The structure of the proofs for the individual sequents is by induction.
- As is sometimes the case with induction, we have to prove a generalization of the main objective.
- In this case we have to prove something for formulas in general, not just tautologies.
- Consider **any combination** $p^*_1, p^*_2, \ldots, p^*_k$ of the symbols negated or un-negated (e.g. $\neg p_1, p_2, \ldots, \neg p_k$) and the corresponding assignment $\alpha(p^*_1) = \alpha(p^*_2) = \ldots \wedge \alpha(p^*_k) = T$. We intend to show:
  - If $\alpha(\eta) = T$ then $p^*_1, p^*_2, \ldots, p^*_k \vdash \eta$.
  - If $\alpha(\eta) = F$ then $p^*_1, p^*_2, \ldots, p^*_k \vdash (\neg\eta)$.
- Tautologies are a special case, where $\alpha(\eta) = T$.

Proving (for any α)

1. If $\alpha(\eta)$ = T then $p^*_1, p^*_2, \ldots, p^*_k \vdash \eta$.

2. If $\alpha(\eta)$ = F then $p^*_1, p^*_2, \ldots, p^*_k \vdash (\neg\eta)$.

- This is done by **structural induction** on the formula η, i.e. as determined by the **grammar** for formulas.

- **Basis**:
  - If η is a single proposition, say p, then
    - If $\alpha(p)$ = T, then p* must be p, and we have $p \vdash p$ in one step because p is a premise.
    - If $\alpha(p)$ = F , then p* must be $\neg p$, and we have $\neg p \vdash \neg p$ in one step (noting that $\neg\eta$ is $\neg p$).
  - If η is $\perp$, then $\alpha(\perp)$ = F. We have $\perp \vdash \neg\perp$ from the $\perp$e rule.

- **Induction Step**:

  We have to show that the inductive hypothesis implies the conclusion for each possible operator: $\neg \wedge \vee \rightarrow$.

Case where η is of form $\neg\rho$:

- If $\alpha(\eta)$ = T, then $\alpha(\rho)$ = F. By the induction hypothesis, part 2:

  $$p^*_1, p^*_2, \ldots, p^*_k \vdash (\neg\rho),$$

  therefore

  $$p^*_1, p^*_2, \ldots, p^*_k \vdash \eta,$$ which is statement 1.

- If $\alpha(\eta)$ = F, then $\alpha(\rho)$ = T. By the induction hypothesis, part 1:

  $$p^*_1, p^*_2, \ldots, p^*_k \vdash \rho.$$

  Using the $\neg\neg$i rule to extend the proof one step, we have

  $$p^*_1, p^*_2, \ldots, p^*_k \vdash \neg(\neg\rho).$$

  Therefore

  $$p^*_1, p^*_2, \ldots, p^*_k \vdash \neg\eta,$$ which is statement 2.

Case where η is of form $\rho_1 \wedge \rho_2$

- Suppose $\alpha(\rho_1 \wedge \rho_2) = T$.
- We must establish
  $$p^*_1, p^*_2, \ldots, p^*_k \vdash (\rho_1 \wedge \rho_2)$$
- Since $\alpha(\rho_1 \wedge \rho_2) = T$, $\alpha(\rho_1) = T$ and $\alpha(\rho_2) = T$.
- Then by the induction hypothesis
  $$p^*_1, p^*_2, \ldots, p^*_k \vdash \rho_1$$
  $$p^*_1, p^*_2, \ldots, p^*_k \vdash \rho_2.$$
  Using $\wedge$i, we get a proof of
  $$p^*_1, p^*_2, \ldots, p^*_k \vdash (\rho_1 \wedge \rho_2)$$
  as desired.

Case where η is of form $\rho_1 \wedge \rho_2$ continued

- Suppose $\alpha(\rho_1 \wedge \rho_2) = F$.
- We must then establish
  $p^*_1, p^*_2, \ldots, p^*_k \vdash \neg(\rho_1 \wedge \rho_2)$
- In this case $\alpha(\rho_1) = F$ **or** $\alpha(\rho_2) = F$.
- By the induction hypothesis
  $p^*_1, p^*_2, \ldots, p^*_k \vdash (\neg\rho_1)$
  or $p^*_1, p^*_2, \ldots, p^*_k \vdash (\neg\rho_2)$.
- In either case, using the appropriate $\vee$i we get
  $p^*_1, p^*_2, \ldots, p^*_k \vdash (\neg\rho_1) \vee (\neg\rho_2)$.
- Then using DeMorgan's rule (a derived rule), we get
  $p^*_1, p^*_2, \ldots, p^*_k \vdash \neg(\rho_1 \wedge \rho_2)$
  as desired.

Case where η is of form $\rho_1 \vee \rho_2$

- Suppose $\alpha(\rho_1 \vee \rho_2) = T$.
- We must establish

  $$p^*_1, p^*_2, \ldots, p^*_k \vdash (\rho_1 \vee \rho_2)$$

- Since $\alpha(\rho_1 \vee \rho_2) = T$, $\alpha(\rho_1) = T$ **or** $\alpha(\rho_2) = T$.

- Then by the induction hypothesis

  $p^*_1, p^*_2, \ldots, p^*_k \vdash \rho_1$

  or $p^*_1, p^*_2, \ldots, p^*_k \vdash \rho_2$.

  Using the appropriate $\vee$i, we get a proof of

  $$p^*_1, p^*_2, \ldots, p^*_k \vdash (\rho_1 \vee \rho_2)$$

  as desired.

Case where η is of form $\rho_1 \vee \rho_2$ continued

- Suppose $\alpha(\rho_1 \vee \rho_2) = F$.
- We must then establish
  $$p^*_1, p^*_2, \ldots, p^*_k \vdash \neg(\rho_1 \vee \rho_2)$$
- In this case $\alpha(\rho_1) = F$ and $\alpha(\rho_2) = F$.
- By the induction hypothesis
  $$p^*_1, p^*_2, \ldots, p^*_k \vdash (\neg\rho_1)$$
  $$p^*_1, p^*_2, \ldots, p^*_k \vdash (\neg\rho_2).$$
- In either case, using $\wedge$i we get
  $$p^*_1, p^*_2, \ldots, p^*_k \vdash (\neg\rho_1) \wedge (\neg\rho_2).$$
- Then using DeMorgan's rule (a derived rule), we get
  $$p^*_1, p^*_2, \ldots, p^*_k \vdash \neg(\rho_1 \vee \rho_2)$$
  as desired.

Case where η is of form $\rho_1 \rightarrow \rho_2$

- Suppose $\alpha(\rho_1 \rightarrow \rho_2) = T$.
- We must then establish
  $p^*_1, p^*_2, \ldots, p^*_k \vdash (\rho_1 \rightarrow \rho_2)$
- In this case $\alpha(\rho_1) = F$ or $\alpha(\rho_2) = T$.
- By the induction hypothesis
  $p^*_1, p^*_2, \ldots, p^*_k \vdash (\neg \rho_1)$
  or $p^*_1, p^*_2, \ldots, p^*_k \vdash \rho_2$.
- In either case, we can derive $\rho_1 \rightarrow \rho_2$ by short derivations.

Case where η is of form $\rho_1 \rightarrow \rho_2$ continued

- Suppose $\alpha(\rho_1 \rightarrow \rho_2) = F$.
- We must then establish
  $p^*_1, p^*_2, \ldots, p^*_k \vdash \neg(\rho_1 \rightarrow \rho_2)$
- In this case $\alpha(\rho_1) = T$ and $\alpha(\rho_2) = F$.
- By the induction hypothesis
  $p^*_1, p^*_2, \ldots, p^*_k \vdash \rho_1$
  and $p^*_1, p^*_2, \ldots, p^*_k \vdash \neg\rho_2$.
- We can then derive $\neg(\rho_1 \rightarrow \rho_2)$ using MP and $\neg$i.

- Thus concludes our proof of the completeness theorem.

Example: $(p \vee \neg q) \to q$

- This is not a tautology.
- There are four assignments possible $\alpha(pq)$ = FF, FT, TF, TT.

- For $\alpha(pq)$ = FF, we have p^* is $\neg p$, q^* is $\neg q$,
  $\alpha((p \vee \neg q) \to q)) = F$.
- So we must establish
  $$\neg p, \neg q \vdash \neg((p \vee \neg q) \to q))$$
- This breaks down into
  $$\neg p, \neg q \vdash (p \vee \neg q)$$
  and $\neg p, \neg q \vdash \neg q$.
- The first one follows from
  $$\neg p, \neg q \vdash \neg q$$
  using $\vee$i. The second one is immediate.

Example: $(p \vee \neg q) \rightarrow q$ continued

- For $\alpha(pq) = FT$, we have p^* is $\neg p$, q^* is q, $\alpha((p \vee \neg q) \rightarrow q)) = T$.

- So we must establish
  $$\neg p, q \vdash ((p \vee \neg q) \rightarrow q))$$
- We must have either
  $$\neg p, q \vdash \neg(p \vee \neg q)$$
  or $$\neg p, q \vdash q.$$

- Use the second one.

Example: $(p \vee \neg q) \rightarrow q$ continued

- For $\alpha(pq) = TF$, we have p^* is p, q^* is $\neg q$,
  $\alpha((p \vee \neg q) \rightarrow q)) = F$.

- So we must establish
  $$p, \neg q \vdash \neg((p \vee \neg q) \rightarrow q))$$

- This can be done if we can first establish both
  $$p, \neg q \vdash (p \vee \neg q)$$
  and $p, \neg q \vdash \neg q$
  both of which can be derived readily.

Example: $(p \vee \neg q) \rightarrow q$ concluded

- For $\alpha(pq)$ = TT, we have p^* is p, q^* is q, $\alpha((p \vee \neg q) \rightarrow q))$ = T.

- So we must establish
  $$p, q \vdash ((p \vee \neg q) \rightarrow q))$$

- This can be done by if we can derive either
  $$p, q \vdash \neg(p \vee \neg q)$$
  or $$p, q \vdash q$$

  Clearly the second is derivable.