Partial Solutions to Sample Exam Questions

1. [20 points]

Consider the following language categories:

4: Finite language
3: Regular language, but not finite
2: Context-Free Language, but not regular
1: Not context-free

For each language in the following table, indicate the category in which the language belongs.

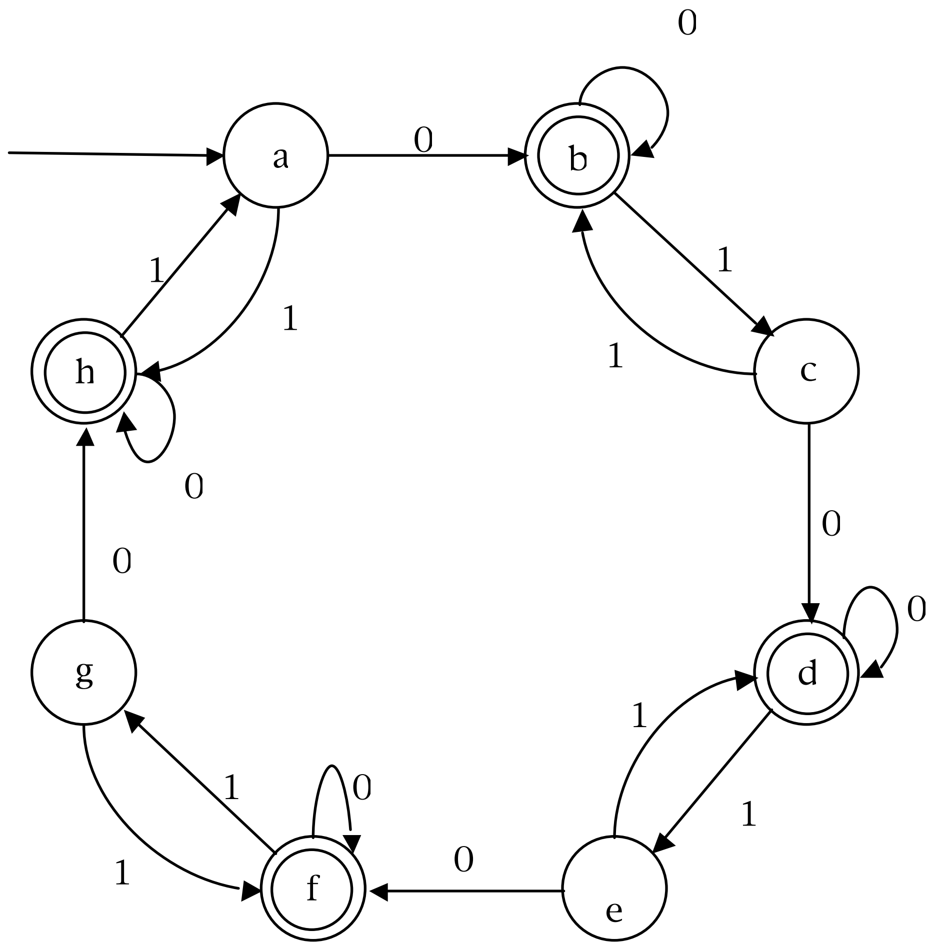
	Language	Category
a.	$\{0^n 1^n 0^n 1^n \mid n \in \omega\}$	1
b.	$\{0^n 1^n \mid n \in \omega\}\{0^n 1^n \mid n \in \omega\}$	2
c.	$\{1^{1000n} \mid n \in \omega\}$	3
d.	$\{0^{1000}, 1^{1000}\}$	4
e.	$\{0\}^*\{0^n 1^n \mid n \in \omega\}\{1\}^*$	3
f.	$\{0^n 1^n \mid n \in \omega\}^*$	2
g.	$\{0^n 1 0^n \mid n \in \omega\}$	2
h.	$\{0^n 0^n \mid n \in \omega\}$	3
i.	$\{xxx \mid x \in \{1\}^+\}$	3
j.	$\{xy \mid x, y \in \{0, 1\}^+, \lvert x\rvert = \lvert y\rvert\}$	3

3. [10 points]

Give a regular expression for the language accepted by the following DFA:

Answer: This DFA can be minimized to one with just two states, as the final partition is {{a, c, e, g}, {b, d, f, h}}. Then the regular expression can be read off as

(0 + 1)(0 + 1 (0 + 1))*.



4. [10 points]

The "Regular Expression Substitution Principle" says that for any regular expression equality we can substitute entire languages in place of the letters and the equality remains valid.

For example, given the equality (0*1)* = (1*0)*, we infer the equality (L*M)* = (M*L)* for any languages L and M.

Assuming that this principle is correct, describe an algorithm for testing the validity of equalities for arbitrary languages, where the equalities involve only the operators union, language product, and star.

Answer: Construct a DFA for each side of the equality, as if the language symbols were letters in the alphabet. Check the result for equivalence using the product construction. Specifically, in the product machine, each state should have both components of the original machine accepting or both non-accepting.

6. [10 points]

Give the transition diagram for a pushdown acceptor that accepts the language

$\{x \in \{a, b, c\}^* \mid \#_a(x) = \#_b(x) + \#_c(x)\}$

For example, abac is in the language, but abc is not.

Answer: The pda accepts by empty-stack. There is only one control state, so it will be omitted. The initial stack symbol is ⊥. The top of stack is shown at the left.

⊥, ε→ ε
⊥, a→ a⊥
⊥, b→ b⊥
⊥, c→ b⊥
a, a→ aa
a, b→ ε
a, c→ ε
b, a→ ε
b, b→ bb
b, c→ bb

8. [10 points]

Use the pumping lemma to show that the language

$$\{a^i b^{i*i} \mid k \in \omega\} = \{\Lambda, ab, aabbbb, aaabbbbbbbbb, ... \}$$

is not context-free.

Answer: Let L be the language and k be pump(L).

Consider $a^k b^{k*k} \in L$. According to the pumping lemma, there are strings u, v, w, x, y such that

I. $uvwxy = a^k b^{k*k}$

II. $| v w x | \leq k$

III. $| v x | > 0$

IV. $\forall i \; uv^i wx^i y \in L$.

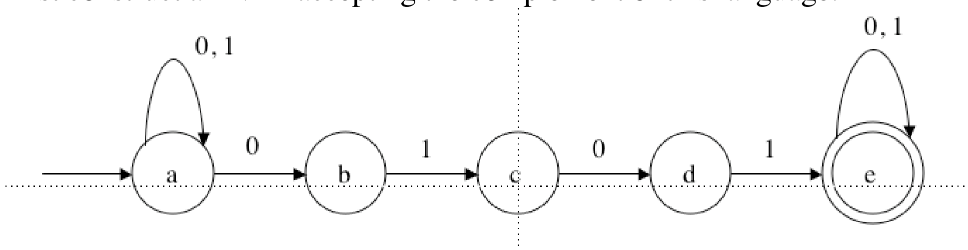
For IV to be true we must have $v \in \{a\}^*$ and $x \in \{b\}^*$due to the form of L.

But clearly not $\forall i \; uv^i wx^i y \in L$, because the number of a's and b's increase linearly with i, but the number of b's would have to increase quadratically with i.

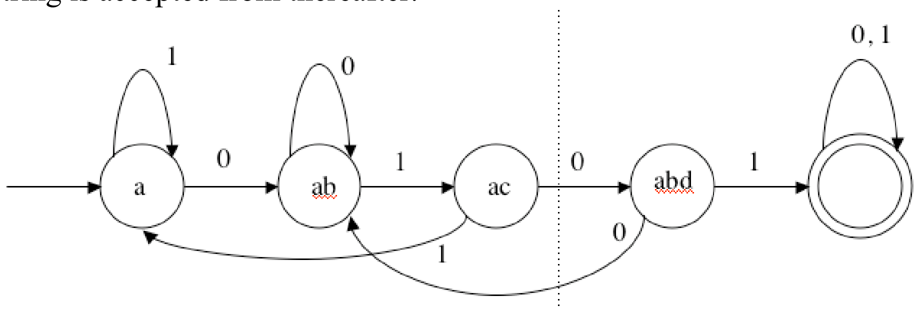
2. [10 points]

Construct a minimum-state deterministic finite-state acceptor that accepts all strings in {0, 1}* not containing 0101.

First construct an NFA accepting the complement of this language:



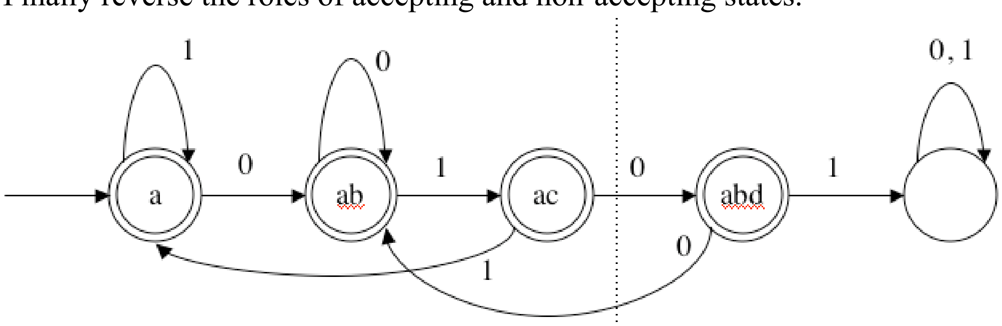
Convert the NFA to a DFA. The accepting state represents the agglomeration of a number of equivalent accepting states. Once 0101 has been recognized in the input, the string is accepted from thereafter.



Check that no two states are equivalent: From each pair of non-accepting states there is some string that distinguishes the states in that pair.

distinguishing	a	ab	ac
a			
ab	101		
ac	01	01	
abd	1	1	1

Finally reverse the roles of accepting and non-accepting states.



1. Classify these formulas as one of {tautology, satisfiable but not a tautology, not satisfiable}:

    a. $((p \to q) \to (q \to r)) \to (p \to r)$ satisfiable, not a tautology
    b. $(p \to q) \to (q \to p)$ satisfiable, not a tautology
    c. $\neg(\neg q \to p)) \to \neg(p \wedge \neg q)$ tautology
    d. $(p \wedge (q \vee r)) \leftrightarrow ((p \wedge q) \vee (p \wedge r))$ tautology

2. For each formula above:

    a. If the formula is a tautology, give a natural deduction proof of it.
    b. If the formula is unsatisfiable, give a natural deduction proof of its negation.
    c. If the formula is satisfiable, but not a tautology, give an assignment that satisfies it and one that does not.

$((p \to q) \to (q \to r)) \to (p \to r)$ satisfiable, not a tautology
satisfies: $v(p) = v(q) = 0$, $v(r) = 1$
not satisfies: $v(p) = 1$, $v(q) = 0$, $v(r) = 0$

$(p \to q) \to (q \to p)$ satisfiable, not a tautology
satisfies: $v(p) = v(q) = 0$
not satisfies: $v(p) = 0$, $v(q) = 1$

$\neg(\neg q \to p)) \to \neg(p \wedge \neg q)$ tautology

1.	$\neg(\neg q \to p))$	Assumption
2.	$p \wedge \neg q$	Assumption
3.	p	$\wedge$E 2
4.	$\neg q \to p$	$\to$I 3
5.	$\bot$	$\neg$E 1, 4
6.	$\neg(p \wedge \neg q)$	$\neg$I 2-5
7.	$\neg(\neg q \to p)) \to \neg(p \wedge \neg q)$	$\to$I 1-6

$(p \wedge (q \vee r)) \leftrightarrow ((p \wedge q) \vee (p \wedge r))$ tautology

1.	$(p \wedge (q \vee r))$	Assumption
2.	p	$\wedge$E 1
3.	$q \vee r$	$\wedge$E 1
4.	q	Assumption
5.	$p \wedge q$	$\wedge$I 2, 4
6.	$(p \wedge q) \vee (p \wedge r)$	$\vee$I 5
7.	r	Assumption
8.	$p \wedge r$	$\wedge$I 2, 6
9.	$(p \wedge q) \vee (p \wedge r)$	$\vee$I 8
10.	$(p \wedge q) \vee (p \wedge r)$	$\vee$E 3, 4-6, 7-9
11.	$(p \wedge q) \vee (p \wedge r)$	Assumption
12.	$p \wedge q$	Assumption
13.	p	$\wedge$E 12
14.	q	$\wedge$E 12
15.	$q \vee r$	$\vee$I 14
16.	$p \wedge (q \vee r)$	$\wedge$I 13, 15
17.	$p \wedge r$	Assumption
18.	p	$\wedge$E 17
19.	r	$\wedge$E 17
20.	$q \vee r$	$\vee$I 19
21.	$p \wedge (q \vee r)$	$\wedge$I 18, 20
22.	$p \wedge (q \vee r)$	$\vee$E 11, 12-16, 17-21
23.	$(p \wedge (q \vee r)) \leftrightarrow ((p \wedge q) \vee (p \wedge r))$	$\leftrightarrow$I 1-10, 11-22

Using natural deduction, derive the rule known as *modus tollendo ponens* from the basic rules given in the text:

$$\frac{\varphi \vee \psi, \qquad \neg\varphi}{\psi}$$

Proof:

$$\dfrac{\dfrac{\dfrac{[\varphi]_1 \quad \neg\varphi}{\bot}\,(\neg E)}{\psi}\,(\bot) \qquad [\psi]_2 \qquad \varphi \vee \psi}{\psi}\,(\vee E_{1,\,2})$$