Overview of curves

- interpolating curves
- hermitian splines
- bezier
- b-splines

Drawing Curves

- Sample curve
- Draw line segments between sample points

Representing Curves

How should we represent a curve?
- Flexibility: Can we use the method for a wide range of curves?
- Efficiency: Can we sample it efficiently?
- Usability: Can a user specify it easily?

Complicated Curves

Simple curves connected end-to-end

Simple Curves

How should we represent a simple curve?
- Flexibility
- Efficiency
- Usability
- Boundary constraints: Can we specify continuity (including derivatives) at boundaries?
**Curve Representation**

- Explicit
- Implicit
- Parametric

**Explicit**

The curve is the trace of a function

Example: \( y = \frac{x^2}{4} \)

**Implicit**

The curve is the zero loci of a function

Example: \( f(x,y) = 4y - x^2 \)

**Explicit: flexibility**

Many useful curves cannot be represented by explicit functions

**Implicit: more flexibility**

\( F(x,y) = x^2 + y^2 - r^2 \)

But how could we describe a half circle?
Implicit: Efficiency

How can we find the zero loci of a function $f(x,y)$?

Curve Representation

- Explicit
- Implicit
- Parametric

Parametric

The curve is the range of a function

Example: $x = 2t$, $y = t^2$

Parametric: tradeoffs

- Flexibility: very expressive, easy to specify portions of curves
- Efficiency: easy to find points on curve
- Boundary conditions: easy to specify
- Usability: not intuitive

OK -- I give up!
Parametric: tradeoffs

- Flexibility: very expressive, easy to specify portions of curves
- Efficiency: easy to find points on curve
- Boundary conditions: easy to specify
- Usability: not intuitive without modeling tools!

Parametric cubic polynomials

- Polynomials are expressive and can be efficiently computed
- Lower degree polynomials can’t express non-planar curves
- Higher degree polynomials
  - Wiggle
  - Computationally more expensive

Parametric cubic curves

\((X(t), Y(t), Z(t))\)

- Interpolating
- Hermite
- Catmull-Rom
- Bézier
- B-spline

Interpolating polynomials

Give me a lowest degree polynomial curve through these points:

- - -

Interpolation

- Points: \((x_0, y_0, z_0), (x_1, y_1, z_1), (x_2, y_2, z_2)\)
- Compute: Quadratic polynomials \(x(t), y(t), z(t)\) such that
  \((x(i), y(i), z(i)) = (x_i, y_i, z_i)\) for \(i=0,1,2\)

Exercise

Give me a parametric quadratic curve through the points
\((1,0,2), (1,1,1), (2,-1,3)\)

Let’s do \(x(t)\) together!
Computing $x(t)$

We want a quadratic polynomial $x(t)$ such that:
- $x(0) = 1$
- $x(1) = 1$
- $x(2) = 2$

Step 1

Give me a quadratic polynomial $x(t)$ such that:
- $x(0) = 1$
- $x(1) = 0$
- $x(2) = 0$

Step 2

Give me a quadratic polynomial $x(t)$ such that:
- $x(0) = 0$
- $x(1) = 1$
- $x(2) = 0$

Step 3

Give me a quadratic polynomial $x(t)$ such that:
- $x(0) = 0$
- $x(1) = 0$
- $x(2) = 2$

Step 4

Give me a quadratic polynomial $x(t)$ such that:
- $x(0) = 1$
- $x(1) = 1$
- $x(2) = 2$

Exercise

You do $y(t)$ and $z(t)$! Write your results on the board.
General solution

\[ x(t) = \sum_{i=0}^{n-1} x_i \frac{\prod_{j=0 \to n-1, j \neq i} (t-j)}{\prod_{j=0 \to n-1, j \neq i} (i-j)} \]

Parametric Curves

How should we represent a simple curve?
- Flexibility
- Efficiency
- Usability

- Boundary constraints: Can we specify continuity (including derivatives) at boundaries?

Parametric cubic curves

- Interpolating
- Hermitian
- Catmull-Rom
- Bezier
- B-spline

Hermitian splines

- Specify endpoint position
- Specify derivative at endpoint

Hermitian

- \( X(t) = at^3 + bt^2 + ct + d \)
- \( X(0) = 3, \ X(1) = 2 \)
- \( X'(0) = 1, \ X'(1) = 0 \)
- Write 4 equations that determine the coefficients \( a, b, c, \) and \( d. \)
Hermitian: Constraints

• X(0):  d = 3
• X(1):  a+b+c+d = 2
• X'(0):  c = 1
• X'(1):  3a+2b+c=0

Hermitian Matrix Form

\[
\begin{pmatrix}
0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 \\
3 & 2 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c \\
d
\end{pmatrix}
= \begin{pmatrix}
3 \\
2 \\
1 \\
0
\end{pmatrix}
\]

Hermitian Matrix Form

\[
\begin{pmatrix}
2 & -2 & 1 & 1 \\
-3 & 3 & -2 & -1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 \\
3 & 2 & 1 & 0
\end{pmatrix}
= ?
\]

find X(t) for this example

Hint

\[
\begin{pmatrix}
2 & -2 & 1 & 1 \\
-3 & 3 & -2 & -1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 \\
3 & 2 & 1 & 0
\end{pmatrix}
= ?
\]

X(t)

\[
\begin{pmatrix}
a \\
b \\
c \\
d
\end{pmatrix}
= \begin{pmatrix}
2 & -2 & 1 & 1 \\
-3 & 3 & -2 & -1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
3 \\
2 \\
1 \\
0
\end{pmatrix}
= \begin{pmatrix}
3 \\
-5 \\
1 \\
3
\end{pmatrix}
\]

X(t) = 3t^3 - 5t^2 + t + 3

Verify

X(t) = 3t^3 - 5t^2 + t + 3

• X(0) = 3,  X(1) = 2
• X'(0) = 1,  X'(1) = 0
General Solution: \( X(t) \)
\[
x(t) = at^3 + bt^2 + ct + d
\]
\[
\begin{bmatrix}
a \\
b \\
c \\
d \\
\end{bmatrix} = \begin{bmatrix}
2 & -2 & 1 & 1 \\
-3 & 3 & -2 & -1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix}
X(0) \\
X(1) \\
X(0) \\
X(1) \\
\end{bmatrix}
\]

General Solution: \( Y(t) \)
\[
Y(t) = at^3 + bt^2 + ct + d
\]
\[
\begin{bmatrix}
a \\
b \\
c \\
d \\
\end{bmatrix} = \begin{bmatrix}
2 & -2 & 1 & 1 \\
-3 & 3 & -2 & -1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix}
Y(0) \\
Y(1) \\
Y(0) \\
Y(1) \\
\end{bmatrix}
\]

General Solution: \( Z(t) \)
\[
Z(t) = at^3 + bt^2 + ct + d
\]
\[
\begin{bmatrix}
a \\
b \\
c \\
d \\
\end{bmatrix} = \begin{bmatrix}
2 & -2 & 1 & 1 \\
-3 & 3 & -2 & -1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix}
Z(0) \\
Z(1) \\
Z(0) \\
Z(1) \\
\end{bmatrix}
\]

Hermitian Basis Matrix
\[
\begin{bmatrix}
2 & -2 & 1 & 1 \\
-3 & 3 & -2 & -1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Exercise
- \( X(0) = 3, \; X(1) = 2, \; X'(0) = 1, \; X'(1) = 0 \)
- \( Y(0) = 2, \; Y(1) = 2, \; Y'(0) = 0, \; Y'(1) = 1 \)

- Write the equations
- Plot the curve for \( t \) in \([0,1]\)

Equations
- \( X(0) = 3, \; X(1) = 2, \; X'(0) = 1, \; X'(1) = 0 \)
- \( Y(0) = 2, \; Y(1) = 2, \; X'(0) = 0, \; X'(1) = 1 \)

- Equations:
  - \( X(t) = 3t^3 - 5t^2 + t + 3 \)
  - \( Y(t) = t^3 - t^2 + 2 \)
Hermitian Description

• Basis Matrix
• Basis (blending) Functions

Hermitian: \( X(t) \)

\[
X(t) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}
\]

General Matrix: \( X \)

\[
\begin{bmatrix}
X(0) \\
X(1) \\
X'(0) \\
X'(1)
\end{bmatrix}
= \begin{bmatrix}
2 & -2 & 1 & 1 \\
-3 & 3 & -2 & -1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

Blending Functions: \( X \)

\[
\begin{bmatrix}
P_1(t) \\
P_2(t) \\
P_3(t) \\
P_4(t)
\end{bmatrix}
= \begin{bmatrix}
X(0) \\
X(1) \\
X'(0) \\
X'(1)
\end{bmatrix}
\]

Hermitian Blending Functions

\[
\begin{bmatrix}
P_1(t) \\
P_2(t) \\
P_3(t) \\
P_4(t)
\end{bmatrix}
= \begin{bmatrix}
2 & -2 & 1 & 1 \\
-3 & 3 & -2 & -1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

Hermitian: problem 1

Specifying derivatives is awkward, particularly when many curves are connected with derivative continuity.
**Parametric Continuity**

\[ C^i : \text{The 0th, 1st, 2nd, ..., } i\text{th derivative of adjacent curves agree at their boundary points.} \]

**Geometric Continuity**

\[ G^0 = C^0 \]

For \( i > 0 \), \( G^i \) means:

- \( G^0 \) continuity plus
- The 1st, 2nd, ..., \( i\)th derivative of adjacent curves are proportional at boundary point.

**Exercise**

- First Hermitian curve \( x(t) \):
  - \( x_1(0), x_1(1), x'_1(0), x'_1(1) \)

- Second Hermitian curve:
  - What conditions provide \( C^i \) & \( G^i \) continuity for \( i=0,1 \)?

**Hermitian: \( G^0 \)**

\[ x_2(0) = x_1(1) \]

**Hermitian: \( G^1 \)**

\[ G^1: x_2(0) = x_1(1) \text{ and } x'_2(0) = \alpha x'_1(1) \text{ for some } \alpha \]

(note: same \( \alpha \) factor applies to \( y(t) \) and \( z(t) \))

**Hermitian: \( C^0 \)**

\[ x_2(0) = x_1(1) \]
Hermitian: $C^1$

\[ x_2(0) = x_1(1) \text{ and } x_2'(0) = x_1'(1) \]

Parametric cubic curves

- Interpolating
- Hermitian
- Catmull-Rom enforces $C^1$ continuity
- Bezier
- B-spline

Catmull-Rom Spline: $C^1$

\[
\begin{aligned}
\text{tangent at } p_1 & = (1/2) < p_2 - p_0 > \\
\text{tangent at } p_2 & = (1/2) < p_3 - p_1 > 
\end{aligned}
\]

Catmull-Rom Basis Matrix

- Compute the basis matrix for the Catmull-Rom spline from $p_i$ to $p_{i+1}$

\[
\begin{pmatrix}
p_i & p_{i+1} & p_{i+2} \\
\end{pmatrix}
\]

Catmull-Rom constraints

\[
X(t) = at^3 + bt^2 + ct + d
\]

- $X(0) = d = x_i$
- $X(1) = a + b + c = x_{i+1}$
- $X'(0) = c = (x_{i+1} - x_{i-1})/2$
- $X'(1) = 3a + 2b + c = (x_{i+2} - x_i)/2$

Catmull-Rom Basis Matrix

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 1 & 0 & c \\
0 & 0 & 1 & 0 \\
3 & 2 & 1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
a \\
b \\
c \\
d \\
\end{pmatrix} =
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & -5 & 0 & 5 \\
-5 & 0 & 5 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
k_{i-1} \\
k_{i} \\
k_{i+1} \\
k_{i+2} \\
\end{pmatrix}
\]
Catmull-Rom Basis Matrix

\[
\begin{array}{cccc}
q & b & c & d \\
1 & 3 & -3 & 1 \\
2 & -5 & 4 & -1 \\
-1 & 0 & 1 & 0 \\
0 & 2 & 0 & 0
\end{array}
\]

\[x_{i+1} = a + 0.5* (b + c) + d \]

Parametric cubic curves

- Interpolating
- Hermite
- Catmull-Rom
- Bezier
- B-spline

Properties of Cubic Bezier Curves

- Control points \( p_0, p_1, p_2, p_3 \)
- Curve starts at \( p_0 \) and ends at \( p_3 \).
- Line segments \( p_0-p_1 \) and \( p_2-p_3 \) are tangent to the curve at, respectively, \( p_0 \) and \( p_3 \).
- The curve lies within the convex hull of the control points.
- Curve is invariant under affine transformations.