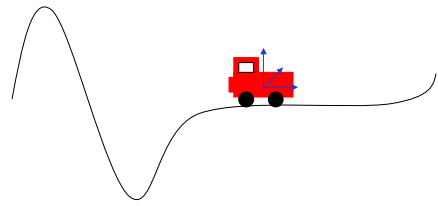


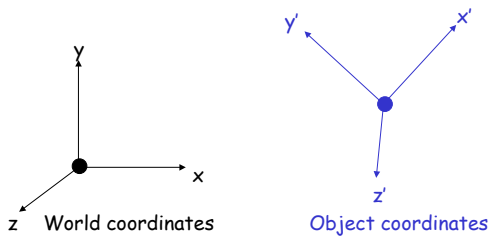
Orientation

Motion along a curve

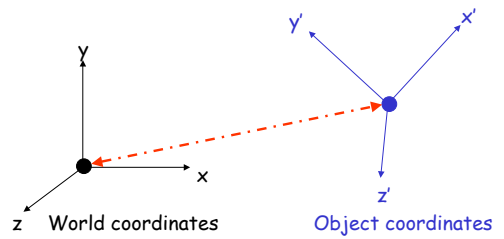


Position + Orientation
(Rigid Body Motion)

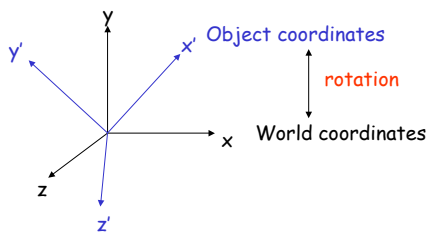
Rigid Body Motion



Position: Translation of origin



Orientation: Rotation of axes



Representation

How should we represent an orientation?

- Rotation matrix
- Rotation angles
- Quaternions

Representation

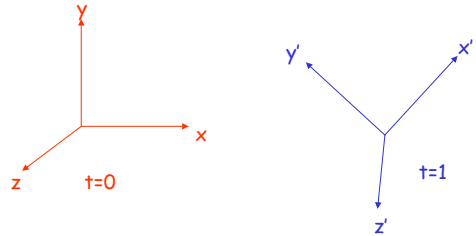
How should we represent an orientation?

Desirable properties:

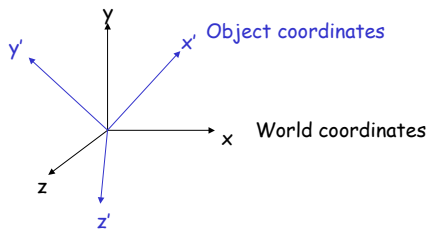
- Able to represent any orientation in a canonical way
- Intuitive
- Support interpolation

Interpolating Orientation

What is the orientation at $t=1/2$?



Rotation Matrix Representation



Euler Theorem

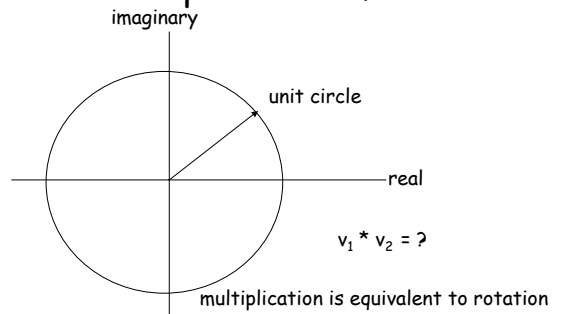
Given a starting orientation S and ending orientation T , there is a single rotation that transforms S to T .

Representation

How should we represent an orientation?

- Rotation matrix
- Rotation angles
- **Quaternions**

Complex Polar form



Rotations

Can we represent a 3D orientation by some extension of complex numbers?

Quaternion Intro

$$q = q_0 + q_1i + q_2j + q_3k$$

multiplication
(non-abelian)

*	i	j	k
i	-1	k	-j
j	-k	-1	i
k	j	-i	-1

Product

$q * q'$: expand and multiply

Shorthand

$q = \langle s, v \rangle$ where s is a scalar (the real part) and v is a vector (the imaginary part)

Shorthand

if $q = \langle s, v \rangle$ and $q' = \langle s', v' \rangle$ then

$$q * q' = \langle ss' - v \cdot v', s'v + s'v + v \times v' \rangle$$

Magnitude squared

$$\|q\|^2 = ss + v \cdot v$$

Complex Conjugate

$$\bar{q} = \langle s, -v \rangle$$

Dot product

$$q \bullet q' = ss' + v \bullet v' = \|q\| \|q'\| \cos(\beta) \text{ where } \beta \text{ is the angle between them}$$

Inverse

Suppose q is a unit quaternion.
What is the inverse of q ?

Hint try the conjugate!

Inverse

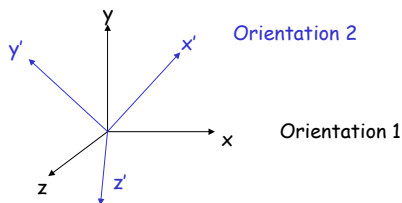
Suppose q is a unit quaternion.
What is the inverse of q ?

Hint try the conjugate!

What if q is not a unit quaternion but rather $\|q\|=M$?

Rotations

A rotation consists of an angle θ and the vector of rotation v . (Euler theorem)



Quaternions and Rotations

We can "encode" the rotation by θ about the unit vector $v = \langle v_1, v_2, v_3 \rangle$ as the quaternion:

$$q = [\cos \theta/2, v_1 \sin \theta/2, v_2 \sin \theta/2, v_3 \sin \theta/2]$$

Quaternions and Rotation

$$q = [\cos \theta/2, v_1 \sin \theta/2, v_2 \sin \theta/2, v_3 \sin \theta/2]$$

What is the magnitude of q ?

Quaternions and Rotations

What is $s*q$ where

$$q = [\cos \theta/2, v_1 \sin \theta/2, v_2 \sin \theta/2, v_3 \sin \theta/2]$$

and

$$s = [\cos \alpha/2, u_1 \sin \alpha/2, u_2 \sin \theta/2, u_3 \sin \alpha/2]$$

Advice

Remember your half angle trig formulas!

Quaternions and Rotations

More generally, the encoding of the rotation q followed by the rotation s is $s*q$ where:

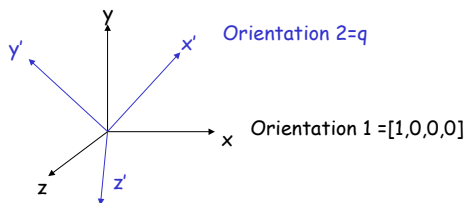
$$q = [\cos \theta/2, v_1 \sin \theta/2, v_2 \sin \theta/2, v_3 \sin \theta/2]$$

and

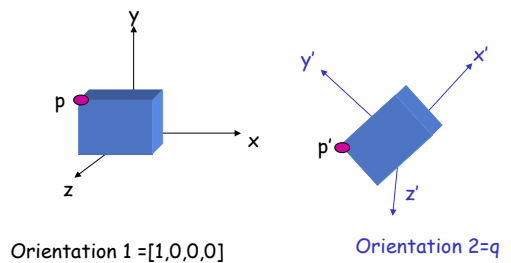
$$s = [\cos \alpha/2, u_1 \sin \alpha/2, u_2 \sin \theta/2, u_3 \sin \alpha/2]$$

(Remember v and u are unit vectors!)

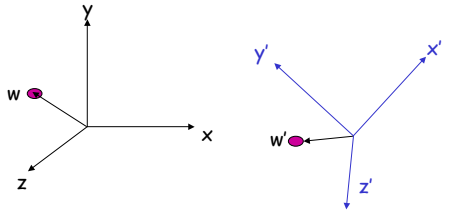
Using quaternions



Using quaternions



Using quaternions

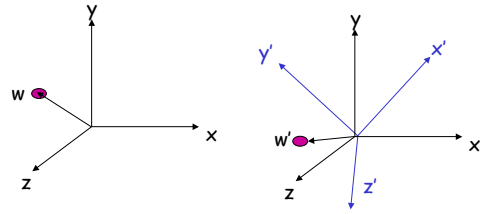


Orientation 1 = [1,0,0,0]

Orientation 2 = q

Using quaternions

What is w' relative to world coordinates?

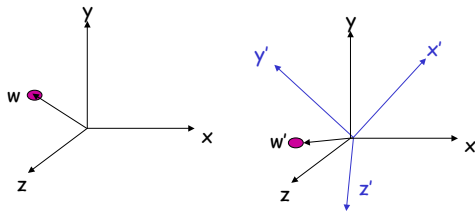


Orientation 1 = [1,0,0,0]

Orientation 2 = q

Using quaternions

$$[0, w'] = q[0, w]q^{-1}$$

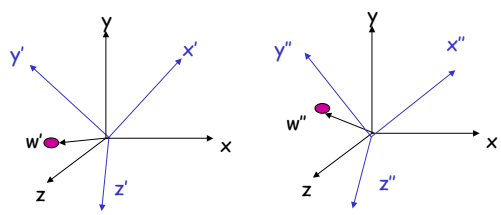


Orientation 1 = [1,0,0,0]

Orientation 2 = q

Using quaternions

Now apply an additional rotation of α about the unit vector u.

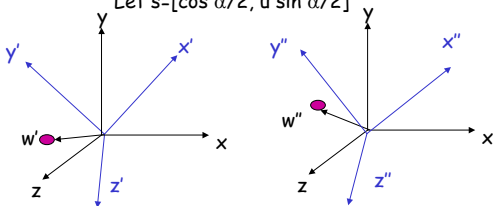


Orientation 2 = q

Using quaternions

Now apply an additional rotation of α about the unit vector u.

$$\text{Let } s = [\cos \alpha/2, u \sin \alpha/2]$$



Orientation 2 = q

Orientation 3 = qs

$$w'' = s(q[0, w]q^{-1})s^{-1} = (sq)[0, w](sq)^{-1}$$

Drawing in the pipeline

Quaternion orientation \rightarrow Rotation Matrix

$$q = [q_0, q_1, q_2, q_3]$$

$$\begin{bmatrix} 1 - 2q_2^2 - 2q_3^2 & 2q_1q_2 + 2q_0q_3 & 2q_1q_3 - 2q_0q_2 \\ 2q_1q_2 - 2q_0q_3 & 1 - 2q_1^2 - 2q_3^2 & 2q_2q_3 + 2q_0q_1 \\ 2q_1q_3 + 2q_0q_2 & 2q_2q_3 - 2q_0q_1 & 1 - 2q_1^2 - 2q_2^2 \end{bmatrix}$$

Next time

slerping: quaternion interpolation