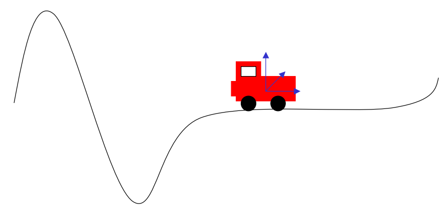
Orientation III

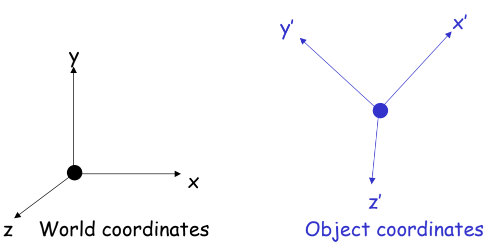
Motion along a curve



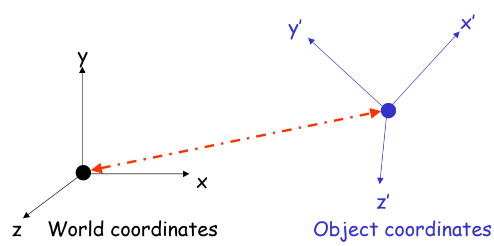
Position + Orientation

(Rigid Body Motion)

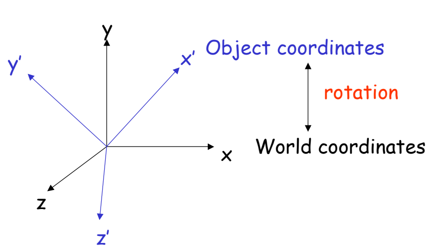
Rigid Body Motion



Position: Translation of origin



Orientation: Rotation of axes



Representation

How should we represent an orientation?

- Rotation matrix
- Rotation angles
- **Quaternions**

Quaternion

$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$

multiplication
(non-abelian)

*	i	j	k
i	-1	k	-j
j	-k	-1	i
k	j	-i	-1

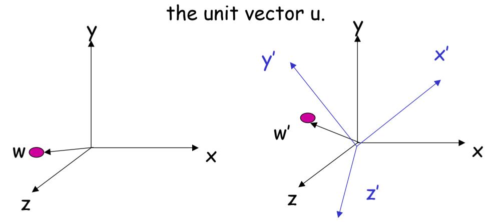
Quanternions and Rotations

We can "encode" the rotation by θ about the unit vector $v = \langle v_1, v_2, v_3 \rangle$ as the quaternion:

$q = [\cos(\theta/2), v \sin(\theta/2)]$
$= [\cos(\theta/2), v_1 \sin(\theta/2), v_2 \sin(\theta/2), v_3 \sin(\theta/2)]$

Using quaternions

Rotate α about
the unit vector u.



$[0,w'] = s[0,w]s^{-1}$

where $s = [\cos a/2, u \sin a/2]$

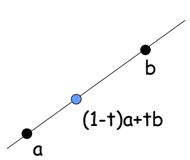
Drawing in the pipeline

Quaternion orientation $\rightarrow$ Rotation Matrix

$q = [q_0, q_1, q_2, q_3]$

$$\begin{bmatrix} 1-2q_2^2-2q_3^2 & 2q_1q_2+2q_0q_3 & 2q_1q_3-2q_0q_2 \\ 2q_1q_2-2q_0q_3 & 1-2q_1^2-2q_3^2 & 2q_2q_3+2q_0q_1 \\ 2q_1q_3+2q_0q_2 & 2q_2q_3-2q_0q_1 & 1-2q_1^2-2q_2^2 \end{bmatrix}$$

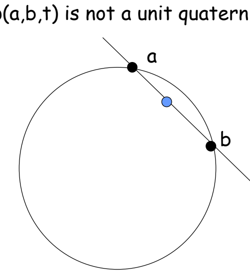
Interpolation



Linear interpolation
lerp(a,b,t)

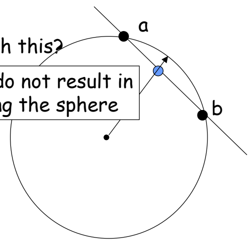
Lerping quaternions

Lerp(a,b,t) is not a unit quaternion



Lerping + normalizing quaternions

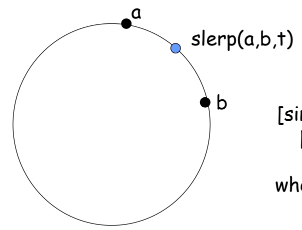
Lerp(a,b,t) then normalize is a unit quaternion.



What is wrong with this?

Uniform steps in t do not result in uniform steps along the sphere

Slerping quaternions

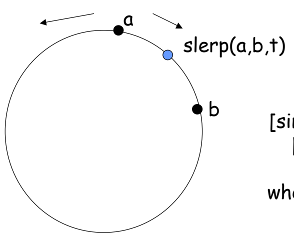


$\text{slerp}(a,b,t) = [\sin((1-t)\theta)/\sin\theta]a + [\sin(t\theta)/\sin\theta]b$

where $\theta = \cos^{-1}(a \bullet b)$

Slerping quaternions

Caveat 1: Which way?



$\text{slerp}(a,b,t) = [\sin((1-t)\theta)/\sin\theta]a + [\sin(t\theta)/\sin\theta]b$

where $\theta = \cos^{-1}(a \bullet b)$

Momentary aside...

What is the difference between

rotation of θ about v and rotation $-\theta$ about $-v$?

Nothing

Momentary aside...

What is the difference between

rotation of θ about v and rotation $2\pi-\theta$ about $-v$?

Nothing

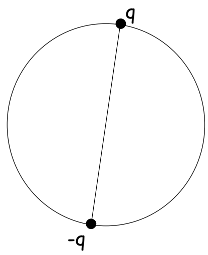
Momentary aside cont.

How does q, which is an encoding of θ about v, compare to the encoding of $2\pi-\theta$ about $-v$?

Ans. $[\cos((2\pi-\theta)/2), -v\sin((2\pi-\theta)/2)]$
$=[\cos(\pi-(\theta/2)), -v\sin(\pi-(\theta/2))]$
$=[-\cos(\theta/2), -v\sin(\theta/2)]$
$= -q$

Note that $q \neq -q$

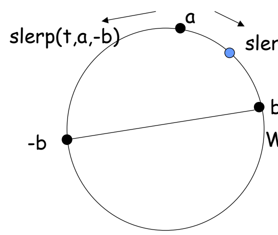
Redundancy



The same orientation can
be represented
by q or -q

Slerping quaternions

Caveat 1: Which way?

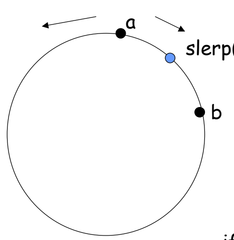


What is the difference between
slerp(a,b,t) and slerp(-a,b,t)?

Ans: one is the long way and one is the short way

Slerping quaternions

Caveat 1: Which way?



slerp(a,b,t) =
$[\sin((1-t)\theta)/\sin\theta]a$ +
$[\sin(t\theta)/\sin\theta]b$

where $\theta = \cos^{-1}(a \bullet b)$

if $(a \bullet b) < 0$, negate one of them
to interpolate the short way

Quaternions

Caveat 2

Is slerp(a,b,t) a unit quaternion?

No!

So project onto unit hypersphere by normalizing!