Orientation III

Motion along a curve
Position + Orientation
(Rigid Body Motion)

Rigid Body Motion

Position: Translation of origin

Orientation: Rotation of axes

Representation

How should we represent an orientation?
- Rotation matrix
- Rotation angles
- Quaternions
**Quaternion**

\[ q = q_0 + q_1i + q_2j + q_3k \]

Multiplication (non-abelian)

\[
\begin{array}{c|cccc}
* & i & j & k & \text{identity} \\
\hline
i & -1 & k & -j & \text{identity} \\
j & -k & -1 & i & \text{identity} \\
k & j & i & -1 & \text{identity}
\end{array}
\]

**Quaternions and Rotations**

We can "encode" the rotation by \( \theta \) about the unit vector \( \mathbf{v} = <v_1,v_2,v_3> \) as the quaternion:

\[ q = \left[ \cos \left( \frac{\theta}{2} \right), v \sin \left( \frac{\theta}{2} \right) \right] \]

\[ = \left[ \cos \left( \frac{\theta}{2} \right), v_1 \sin \left( \frac{\theta}{2} \right), v_2 \sin \left( \frac{\theta}{2} \right), v_3 \sin \left( \frac{\theta}{2} \right) \right] \]

**Using quaternions**

Rotate \( \alpha \) about the unit vector \( \mathbf{u} \).

\[ [0, w'] = s [0, w] s^{-1} \]

where \( s = [\cos \alpha/2, \mathbf{u} \sin \alpha/2] \)

**Drawing in the pipeline**

Quaternion orientation \( \rightarrow \) Rotation Matrix

\[
q = [q_0, q_1, q_2, q_3] =
\begin{bmatrix}
1 - 2q_2^2 - 2q_3^2 & 2q_1q_2 - 2q_0q_3 & 2q_1q_3 + 2q_0q_2 \\
2q_1q_2 + 2q_0q_3 & 1 - 2q_1^2 - 2q_3^2 & 2q_2q_3 - 2q_0q_1 \\
2q_1q_3 - 2q_0q_2 & 2q_2q_3 + 2q_0q_1 & 1 - 2q_1^2 - 2q_2^2
\end{bmatrix}
\]

**Interpolation**

Linear interpolation \( \text{lerp}(a,b,t) = (1-t)a + tb \)

**Lerping quaternions**

\( \text{Lerp}(a,b,t) \) is not a unit quaternion
Lerping + normalizing quaternions

Lerp\((a, b, t)\) then normalize is a unit quaternion.

Uniform steps in \(t\) do not result in uniform steps along the sphere.

What is wrong with this?

Slerping quaternions

\[
\text{slerp}(a, b, t) = \left[ \sin((1-t)\theta) / \sin\theta \right] a + \left[ \sin(t\theta) / \sin\theta \right] b
\]

where \(\theta = \cos^{-1}(a \cdot b)\)

Slerping quaternions

Caveat 1: Which way?

\[
\text{slerp}(a, b, t) = \left[ \sin((1-t)\theta) / \sin\theta \right] a + \left[ \sin(t\theta) / \sin\theta \right] b
\]

where \(\theta = \cos^{-1}(a \cdot b)\)

Momentary aside...

What is the difference between rotation of \(\theta\) about \(v\) and rotation \(-\theta\) about \(-v\)?

Nothing

Momentary aside...

What is the difference between rotation of \(\theta\) about \(v\) and rotation \(2\pi - \theta\) about \(-v\)?

Nothing

Momentary aside cont.

How does \(q\), which is an encoding of \(\theta\) about \(v\), compare to the encoding of \(2\pi - \theta\) about \(-v\)?

Ans. \([\cos((2\pi - \theta)/2), -v \sin((2\pi - \theta)/2)]\)

\([\cos((2\pi - \theta)/2), -v \sin((2\pi - \theta)/2)]\)

\([-\cos(\theta/2), -v \sin(\theta/2)]\) = \(-q\)

Note that \(q = -q\)
Redundancy

The same orientation can be represented by \( q \) or \(-q\).

Slerping quaternions

Caveat 1: Which way?

\[ \text{slerp}(t, a, -b) \]

What is the difference between \( \text{slerp}(a, b, t) \) and \( \text{slerp}(-a, b, t) \)?

Ans: one is the long way and one is the short way.

Slerping quaternions

Caveat 1: Which way?

\[ \text{slerp}(a, b, t) = \frac{\sin((1-t)\theta)}{\sin\theta}a + \frac{\sin(t\theta)}{\sin\theta}b \]

where \( \theta = \cos^{-1}(a \cdot b) \)

if \( a \cdot b < 0 \), negate one of them to interpolate the short way.

Quaternions

Caveat 2

Is \( \text{slerp}(a, b, t) \) a unit quaternion?

No!

So project onto unit hypersphere by normalizing!