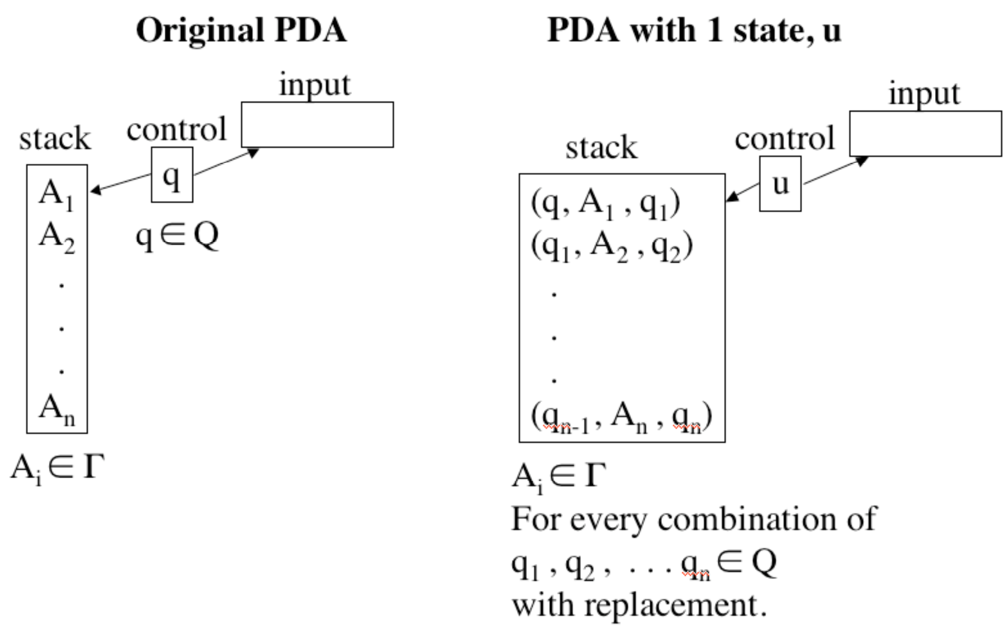
Converting a PDA to an Equivalent 1-state PDA

Robert Keller

February 2007

Basic simulation:

A configuration of the form on the left below is reachable in the original PDA iff every configuration of the form on the right is reachable in the 1-state PDA. In particular, a configuration where the stack is empty on the left will be reachable iff an empty stack configuration is reached on the right.



The words "with replacement" mean that the q_i are not necessarily distinct.

The following are noted about the stack in the 1-state machine:

a. The middle column of symbols is the same as the stack in the original machine.

b. The left-most component of the top triple of the stack corresponds to the control state in the original machine.

c. The right-most component of each triple is identical to the left-most component of the triple below (if any).

The purpose of what might appear to be redundancy in c above is to ensure that b remains true, even when the stack is popped and no symbols are pushed.

The transition rules of the 1-state machine have to be constructed so as to ensure that conditions a-c remain invariant throughout the behavior of the 1-state machine. For each transition of the form

$$q, \sigma, A \rightarrow q', B_1\, B_2 \ldots B_k$$

of the original machine, there is a *set* of transitions for the 1-state machine (where u is the one-and-only state of the latter):

$$u, \sigma, (q, A, q_k), \rightarrow u, (q', B_1, q_1)\, (q_1, B_2, q_2) \ldots (q_{k-1}, B_k, q_k)$$

one transition for each $q_i \in Q$ (with replacement). For $k > 0$, the structure of the right-hand side of the transitions in the set obviously ensure that symbols pushed onto the stack in the 1-state maintain a-c, since the new transitions parallel the original very closely.

What is less clear is the case where $k = 0$, i.e. nothing is pushed on the stack when is popped. The convention is that for $k = 0$, $q_0 = q'$, the next state of the original machine. That is, if

$$q, \sigma, A \rightarrow q', \varepsilon$$

is the transition of the original machine, then there is just one transition for the 1-state machine (where u is the one-and-only state of the latter):

$$u, \sigma, (q, A, q') \rightarrow u, \varepsilon$$

The rationale here is that, when (q, A, q') is effectively popped without pushing anything, the left-most component of the uncovered top of the stack is nothing other than q', so that condition b is maintained.

There is one more thing we need to handle: the initialization of the 1-state machine. For this purpose, define the initial stack symbol in the 1-state machine to be a new symbol x, and add transitions that push every combination of the form (q_0, s, q) for each $q \in Q$

$$u, \varepsilon, x \rightarrow u, (q_0, s, q)$$

where q_0 is the initial state of the original machine and s is the initial stack symbol of the original machine.

Below is an example translation of a 2-state machine accepting the even-length, non-empty, palindromes over {a, b}.

Original PDA	**1-state conversion**
There are two states $\{q_0, q_1\}$. q_0 is the initial state. The stack alphabet is {s, a, b}. s is the initial stack symbol.	There is one state {u}. u is the initial state. The stack alphabet is $\{x\} \cup (\{q_0, q_1\} \times \{s, a, b\} \times \{q_0, q_1\})$. x is the initial stack symbol.
Transitions in original pda	**Transitions in 1-state pda**
	$u, \varepsilon, x \to u, (q_0, s, q_0)$ $u, \varepsilon, x \to u, (q_0, s, q_1)$
$q, \sigma, A \to q', B_1 B_2 \ldots B_k$	$u, \sigma, (q, A, q_k) \to u, (q', B_1, q_1)(q_1, B_2, q_2) \ldots (q_{k-1}, B_k, q_k)$
$q_0, a, s \to q_0, as$	$u, a, (q_0, s, q_0) \to u, (q_0, a, q_0)(q_0, s, q_0)$ $u, a, (q_0, s, q_1) \to u, (q_0, a, q_0)(q_0, s, q_1)$ $u, a, (q_0, s, q_0) \to u, (q_0, a, q_1)(q_1, s, q_0)$ $u, a, (q_0, s, q_1) \to u, (q_0, a, q_1)(q_1, s, q_1)$
$q_0, b, s \to q_0, bs$	$u, b, (q_0, s, q_0) \to u, (q_0, b, q_0)(q_0, s, q_0)$ $u, b, (q_0, s, q_1) \to u, (q_0, b, q_0)(q_0, s, q_1)$ $u, b, (q_0, s, q_0) \to u, (q_0, b, q_1)(q_1, s, q_0)$ $u, b, (q_0, s, q_1) \to u, (q_0, b, q_1)(q_1, s, q_1)$
$q_0, a, a \to q_0, aa$	$u, a, (q_0, a, q_0) \to u, (q_0, a, q_0)(q_0, a, q_0)$ $u, a, (q_0, a, q_1) \to u, (q_0, a, q_0)(q_0, a, q_1)$ $u, a, (q_0, a, q_0) \to u, (q_0, a, q_1)(q_1, a, q_0)$ $u, a, (q_0, a, q_1) \to u, (q_0, a, q_1)(q_1, a, q_1)$
$q_0, b, a \to q_0, ba$	$u, b, (q_0, a, q_0) \to u, (q_0, b, q_0)(q_0, a, q_0)$ $u, b, (q_0, a, q_1) \to u, (q_0, b, q_0)(q_0, a, q_1)$ $u, b, (q_0, a, q_0) \to u, (q_0, b, q_1)(q_1, a, q_0)$ $u, b, (q_0, a, q_1) \to u, (q_0, b, q_1)(q_1, a, q_1)$
$q_0, a, b \to q_0, ab$	$u, a, (q_0, b, q_0) \to u, (q_0, a, q_0)(q_0, b, q_0)$ $u, a, (q_0, b, q_1) \to u, (q_0, a, q_0)(q_0, b, q_1)$ $u, a, (q_0, b, q_0) \to u, (q_0, a, q_1)(q_1, b, q_0)$ $u, a, (q_0, b, q_1) \to u, (q_0, a, q_1)(q_1, b, q_1)$
$q_0, b, b \to q_0, bb$	$u, b, (q_0, b, q_0) \to u, (q_0, b, q_0)(q_0, b, q_0)$ $u, b, (q_0, b, q_1) \to u, (q_0, b, q_0)(q_0, b, q_1)$ $u, b, (q_0, b, q_0) \to u, (q_0, b, q_1)(q_1, b, q_0)$ $u, b, (q_0, b, q_1) \to u, (q_0, b, q_1)(q_1, b, q_1)$
$q_0, \varepsilon, a \to q_1, a$	$u, \varepsilon, (q_0, a, q_0) \to u, (q_1, a, q_0)$ $u, \varepsilon, (q_0, a, q_1) \to u, (q_1, a, q_1)$
$q_0, \varepsilon, b \to q_1, b$	$u, \varepsilon, (q_0, b, q_0) \to u, (q_1, b, q_0)$ $u, \varepsilon, (q_0, b, q_1) \to u, (q_1, b, q_1)$
$q_1, a, a \to q_1, \varepsilon$	$u, a, (q_1, a, q_1) \to u, \varepsilon$
$q_1, b, b \to q_1, \varepsilon$	$u, b, (q_1, b, q_1) \to u, \varepsilon$
$q_1, \varepsilon, s \to q_1, \varepsilon$	$u, \varepsilon, (q_1, s, q_1) \to u, \varepsilon$

Sample accepting sequence, where input is abba:

(state, stack, input)	**(state, stack, input)**
	(u, x, abba),
$(q_0, s, abba)$	$(u, (q_0, s, q_1), abba)$
(q_0, as, bba)	$(u, (q_0, a, q_1)(q_1, s, q_1), bba)$
(q_0, bas, ba)	$(u, (q_0, b, q_1)\ (q_1, a, q_1)(q_1, s, q_1), ba)$
(q_1, bas, ba)	$(u, (q_1, b, q_1)\ (q_1, a, q_1)(q_1, s, q_1), ba)$
(q_1, as, a)	$(u, (q_1, a, q_1)(q_1, s, q_1), a)$
(q_1, s, ε)	$(u, (q_1, s, q_1), \varepsilon)$
$(q_1, \varepsilon,\varepsilon)$	$(u, \varepsilon, \varepsilon)$

Note that there are many non-accepting sequences. These can take the 1-state machine to $(u, (q_1, s, q_0), \varepsilon)$ for example, but there is no transition from there.