Computer Science 81, Spring 2007
Assignment 6
Due Wed. Feb. 28
Proposition Logic

1. [35 points] Using the refutation tree method, determine whether the following formulas are tautologies. For any formula that is not, provide a counterexample in the form of a valuation that induces the value 0.

   a. $((p \rightarrow q) \rightarrow r) \rightarrow (p \rightarrow (q \rightarrow r))$

   b. $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow r)$

   c. $(p \rightarrow (q \rightarrow r)) \rightarrow (\neg r \rightarrow (\neg q \rightarrow \neg p))$

   d. $((p \vee q) \rightarrow (r \vee s)) \rightarrow ((p \rightarrow r) \vee (q \rightarrow s))$

   e. $((p \rightarrow r) \vee (q \rightarrow s)) \rightarrow ((p \vee q) \rightarrow (r \vee s))$

   f. $((p \rightarrow r) \wedge (q \rightarrow s)) \rightarrow ((p \wedge q) \rightarrow (r \wedge s))$

   g. $(p \leftrightarrow (q \leftrightarrow r)) \rightarrow ((p \leftrightarrow q) \leftrightarrow r)$

2. [40 points] Where possible, present natural deduction proofs for the following generic sequents in both the tabular and tree forms. Make note of any that seem to require RAA or its equivalents. If a proof does not seem possible, it is acceptable to present a counterexample as an alternative.

   a. $(\varphi \rightarrow \psi) \vdash ((\varphi \rightarrow \neg\psi) \rightarrow \neg\varphi)$

   b. $((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \xi)) \vdash (\varphi \rightarrow (\psi \rightarrow \xi))$

   c. $(\varphi \wedge \psi) \vdash \neg((\neg\varphi) \vee (\neg\psi))$

   d. $\neg((\neg\varphi) \vee (\neg\psi)) \vdash (\varphi \wedge \psi)$

3. [10 points] **Definition**: Let Γ be a set of formulas (not necessarily finite). By $\Gamma \vdash \psi$ we mean that there is a derivation ψ of that uses as premises only formulas in Γ. Argue that $(\Gamma \cup \{\varphi\}) \vdash \psi$ iff $\Gamma \vdash (\varphi \rightarrow \psi)$.

4. [15 points] Prove by mathematical induction that for any formulas $\varphi_1, \varphi_2, \ldots, \varphi_n$, $n \geq 0$ and any formula ψ that the following are equivalent:
   a. $\{\varphi_1, \varphi_2, \ldots, \varphi_n\} \vdash \psi$
   b. $\vdash \varphi_1 \rightarrow (\varphi_2 \rightarrow (\ldots \rightarrow (\varphi_n \rightarrow \psi) \ldots))$
   c. $\vdash (\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n) \rightarrow \psi$