Computer Science 81, Spring 2007
Assignment 7
Due Wed. Mar. 7
Soundness

1. [50 points] In proving the Soundness Theorem, we used Γ to represent a set that contained the hypotheses (or premises) of a derivation tree $\mathcal{D}$ of a formula φ. Let Hyp($\mathcal{D}$) represent the set of hypotheses *actually used* in derivation $\mathcal{D}$. (Thus Hyp($\mathcal{D}$) $\vdash \varphi$ when $\mathcal{D}$ derives φ.)

   Give an inductive definition of Hyp($\mathcal{D}$) based on the basic natural deduction rules $\{\wedge E, \wedge I, \rightarrow E, \rightarrow I, \bot E, RAA\}$. Note that temporary hypotheses do not count as hypotheses.

   For example, in using the $\wedge I$ rule to combine two derivations $\mathcal{D}, \mathcal{D}'$ into a new one, $\mathcal{D}''$, we have Hyp($\mathcal{D}''$) = Hyp($\mathcal{D}$) $\cup$ Hyp($\mathcal{D}'$). However, in some cases hypotheses are removed, rather than added.

   Demonstrate your definition on the derivation below by showing how Hyp is applied to each of the parts as the proof is built up using the rules. (You will need to identify what those parts are.) The rule names have been left out to avoid clutter, but can be inferred. The numbers remain to show discharged hypotheses. In applying various steps, the nature of a hypothesis can change from ordinary to temporary.

$$\dfrac{\dfrac{\neg(\neg p \wedge \neg q) \qquad \dfrac{\dfrac{\dfrac{\dfrac{p \rightarrow r \quad [p]_1}{r} \quad [\neg r]_3}{\bot}\;{}_1}{\neg p} \qquad \dfrac{\dfrac{\dfrac{q \rightarrow r \quad [q]_2}{r} \quad [\neg r]_3}{\bot}\;{}_2}{\neg q}}{\neg p \wedge \neg q}}{\bot}\;{}_3}{r}$$

2. [20 points] Determine whether each of the following sets of formulas is consistent (meaning that $\bot$ isn't derivable from the set, as discussed in the lecture of 2/28).

a. $\{p \rightarrow q, q \rightarrow r, r \rightarrow p, \neg r\}$	c. $\{p \rightarrow q, q \rightarrow r, r \rightarrow \neg p, r\}$
b. $\{p \rightarrow \neg q, q \rightarrow \neg r, r \rightarrow \neg p\}$	d. $\{p \rightarrow \neg q, \neg q \rightarrow r, r \rightarrow \neg p, p\}$

3. [30 points] By a *literal*, we mean a formula that is either a proposition symbol, or the negation of a proposition symbol. By a *clause*, we mean a disjunction of literals $L_1 \vee L_2 \vee \ldots \vee L_m$. (The disjunction could be empty, in which case we use $\bot$ to represent it.) A formula is in *Conjunctive Normal Form* (yes, another CNF) provided that it has the form $C_1 \wedge C_2 \wedge \ldots \wedge C_n$, where each C_i is a clause. (If the conjunction is empty, we use $\top$ to represent it.) Establish that for every formula φ there is a logically equivalent formula ψ in CNF.