Computer Science 81, Spring 2007
Assignment 8
Due Wed. Mar. 28
Predicate Logic
80 points + 30 extra credit

Recall that

$I \models \varphi$

means that formula φ is valid (true) in interpretation I, while

$\models \varphi$

means $I \models \varphi$ for *every* relevant interpretation I.

1. [20 points] Let I be the interpretation for the formulas below having as its structure the natural numbers, with domain N = {0, 1, 2, 3, …}, one function + (addition), two predicates < (less-than) and = (equality) with their usual meanings, and a constant for every number. In the formulas, symbols 0, +, and < are identified with the corresponding components of the structure. Assess *by evaluation from first principles* whether the following are valid:

   a. [2 points] $I \models \exists y\ (0 < y)$

   b. [3 points] $I \models \forall x\ \exists y\ (x < y)$

   c. [4 points] $I \models \forall x\ \forall y\ ((x < y) \vee (y < x))$

   d. [5 points] $I \models \forall x\ \forall y\ ((x \leq y) \leftrightarrow \exists z\ (x + z = y))$

   e. [6 points] $I \models \exists x\ ((x < 5) \rightarrow \forall x\ (x < 5))$ (This one is tricky.)

2. [20 points] Show the following, assuming that each top-level formula is a sentence (has no free variables):

   a. $\models ((\forall x\ \varphi) \rightarrow (\exists x\ \psi)) \rightarrow (\exists x\ (\varphi \rightarrow \psi))$

   b. $\models ((\exists x\ \varphi) \rightarrow (\forall x\ \psi)) \rightarrow (\forall x\ (\varphi \rightarrow \psi))$

3. [10 points] Show that for any two formulas φ, ψ:

   If for every interpretation I, $I \models \varphi$ implies $I \models \psi$
   then $\models \varphi$ implies $\models \psi$.

4. [10 points] Show that the converse of the statement in the previous problem is *not* true, that is:

   If $\models \varphi$ implies $\models \psi$.
   then for every interpretation I, $I \models \varphi$ implies $I \models \psi$.

5. [20 points] Show by natural deduction:

   a. [10 points] $\vdash (\forall x\, (\varphi \rightarrow \psi)) \rightarrow ((\forall x\, \varphi) \rightarrow (\forall x\, \psi))$

   b. [10 points] $\vdash (\exists x\, (\varphi \rightarrow \psi)) \rightarrow ((\forall x\, \varphi) \rightarrow (\exists x\, \psi))$

6. [Extra credit: 30 points] Show by natural deduction the two formulas in problem 2.