Soundness of Propositional Logic

Recall φ formula
v valuation

$$v(\varphi) \in \{0,1\}$$

v satisfies φ iff $v(\varphi) = 1$.

φ is a tautology iff φ is satisfied by every v:

$$\models \varphi$$

Extension of notation:
Γ is a set of formulas

v satisfies Γ iff v satisfies each $\varphi \in \Gamma$.

$$\Gamma \models \varphi$$

means for each v that satisfies Γ
v also satisfies φ

read Γ entails φ.

Example
$$\{p \to q,\ p \to r,\ p\} \models q \wedge r$$

Parallel notation for derivability

$$\Gamma \vdash \varphi$$

means φ is derivable from premises in Γ.

Example $\{p \to q, p \to r, p\} \vdash q \wedge r$

Proof

$$\dfrac{\dfrac{p \quad p \to q}{q} \to E \qquad \dfrac{p \quad p \to r}{r} \to E}{q \wedge r}$$

Soundness A system of deductive rules is sound iff

for all Γ, φ $\quad \underbrace{\Gamma \vdash \varphi}_{\text{derivability}}$ implies $\underbrace{\Gamma \models \varphi}_{\text{entailment}}$.

Soundness Theorem Our system of rules $\{\wedge I, \wedge E, \to I, \to E, \bot E, RAA\}$ is sound.

Proof Structural induction on derivations is used:

$\Gamma \vdash \varphi$ means there is a derivation

$$\begin{array}{c} D \\ \varphi \end{array} \qquad \text{(} D \text{ is a tree with root } \varphi. \text{)}$$

where all premises used in D are in Γ.

We must show that

whenever $\begin{array}{c} D \\ \varphi \end{array}$ with all premises in Γ

also $\Gamma \models \varphi$.

Basis: If $\begin{array}{c} D \\ \varphi \end{array}$ has just φ as its only formula, then φ must be in Γ.

But $\varphi \in \Gamma$ implies $\Gamma \models \varphi$, trivially.

Induction Steps (one for each rule):

$\wedge$ Introduction rule:

Derivation has form
$$\dfrac{\begin{array}{cc} D & D' \\ \varphi & \varphi' \end{array}}{\varphi \wedge \varphi'} \wedge I$$

Let Γ, Γ' be the premises of D, D' respectively.

Thus $\Gamma \vdash \varphi$, $\Gamma' \vdash \varphi'$.

By the induction hypothesis $\Gamma \models \varphi$, $\Gamma' \models \varphi'$.

Suppose Γ'' contains the premises of $\dfrac{\begin{matrix} D & D' \\ \varphi & \varphi' \end{matrix}}{\varphi \wedge \varphi'}$.

then $\Gamma \cup \Gamma' \subseteq \Gamma''$.

Let v be any valuation satisfying Γ''.

Then v satisfies both Γ and Γ'.

Hence $v(\varphi) = v(\varphi') = 1$ by the induction hypothesis.

Thus $v(\varphi \wedge \varphi') = 1$, so $\Gamma'' \models \varphi \wedge \varphi'$.

<u>$\wedge$ Elimination Rule</u>:

For a derivation $\dfrac{\begin{matrix} D \\ \varphi \wedge \psi \end{matrix}}{\varphi}$

where Γ is the set of premises used in D,

$\Gamma \vdash \varphi \wedge \psi$. By the induction hypothesis

$\Gamma \models \varphi \wedge \psi$. If $v(\varphi \wedge \psi) = 1$, then $v(\varphi) = 1$,
so $\models \varphi$.

$\rightarrow$ Introduction Rule:

$$\dfrac{\begin{matrix}[\varphi] \\ \vdots \\ \psi\end{matrix}}{\varphi \to \psi}$$

Let Γ be the premises of $\begin{matrix}\varphi \\ D \\ \psi\end{matrix}$

By the induction hypothesis $\Gamma \models \psi$.

Let Γ' contain the premises of $\dfrac{\begin{matrix}[\varphi] \\ D \\ \psi\end{matrix}}{\varphi \to \psi}$

Then $\Gamma \subseteq \Gamma' \cup \{\varphi\}$. (Note: φ may be absent from Γ!)

So if v satisfies $\Gamma' \cup \{\varphi\}$ then v satisfies Γ.

By $\Gamma \models \psi$, if v satisfies Γ, v satisfies ψ.

Now suppose v satisfies Γ'. $v(\varphi)$ may be 0 or 1.

If $v(\varphi) = 0$, then $v(\varphi \to \psi) = 1$ vacuously.

If $v(\varphi) = 1$, then v satisfies both Γ' and φ, i.e. v satisfies $\Gamma' \cup \{\varphi\}$.

Hence v satisfies Γ' implies v satisfies ψ,

thus again $v(\varphi \to \psi) = 1$. So $\Gamma' \models \varphi \to \psi$.

$\to$ Elimination Rule :

The derivation is of form

$$\dfrac{\begin{matrix} D & D' \\ \varphi & \varphi \to \psi \end{matrix}}{\psi}$$

Let Γ, Γ' be the premises of D, D', respectively.

The induction hypothesis is $\Gamma \models \varphi, \Gamma' \models \varphi \to \psi$.

Let Γ'' contain the premises of the overall derivation. Then $\Gamma \cup \Gamma' \subseteq \Gamma''$.

Suppose v satisfies Γ''. Then it satisfies Γ and Γ'.

So $v(\varphi) = v(\varphi \to \psi) = 1$. By $H_\to$ we see $v(\psi) = 1$, i.e. v satisfies ψ.

$\perp$ Elimination Rule :

The derivation is of the form

$$\dfrac{\begin{matrix} D \\ \perp \end{matrix}}{\varphi}$$

Let Γ be the premises of $\begin{matrix} D \\ \perp \end{matrix}$.

The induction hypothesis is $\Gamma \models \bot$

which means Γ is satisfied by no v

(since $v(\bot) = 0$ always).

Thus $\Gamma \models \varphi$ (for all v satisfying Γ, i.e. none, v satisfies φ)

If Γ' contains the premises of
$$\begin{array}{c} D \\ \bot \\ \hline \varphi \end{array}$$

it contains those of $\begin{array}{c} D \\ \bot \end{array}$, i.e. $\Gamma \subseteq \Gamma'$.

Suppose v satisfies Γ'. Then it satisfies Γ,

thus φ. Hence $\Gamma' \models \varphi$.

<u>RAA Rule</u>:

The derivation has the form
$$\begin{array}{c} [\neg\varphi] \\ D \\ \bot \\ \hline \varphi \end{array}$$

Let Γ be the premises of
$$\begin{array}{c} \neg\varphi \\ D \\ \bot \end{array}$$

By the induction hypothesis $\Gamma \models \bot$.

This says there is no v satisfying Γ.

Now suppose Γ' includes the premises of the overall derivation

$$\dfrac{\begin{array}{c}[\neg\varphi]\\ D\\ \perp\end{array}}{\varphi}$$

then $\Gamma \subseteq \Gamma' \cup \{\neg\varphi\}$.

We want to show $\Gamma' \models \varphi$.

Suppose not, i.e. v is a valuation such that v satisfies Γ' but not φ.

Then $v(\neg\varphi) = 1$. So v satisfies $\Gamma' \cup \{\neg\varphi\}$.

Thus v must satisfy $\Gamma \subseteq \Gamma' \cup \{\neg\varphi\}$.

By the induction hypothesis, that $\Gamma \models \perp$, we have a contradiction.

Implication of Soundness

Soundness: $\Gamma \vdash \varphi$ implies $\Gamma \models \varphi$

Thus not $(\Gamma \models \varphi)$ implies not $(\Gamma \vdash \varphi)$.

In particular, if φ is not a tautology (not $\models \varphi$) it is not derivable (not $\vdash \varphi$).

So we have a <u>necessity</u> test for derivability.

Completeness

A set of rules is complete iff

$$\Gamma \models \varphi \text{ implies } \Gamma \vdash \varphi.$$

In particular, every tautology is derivable.

It is much less obvious how to show this.

Consistency : Derivability = Satisfiability : Truth

Definition Set of formulas Γ is consistent iff not $(\Gamma \vdash \bot)$.

Similary Γ is satisfiable iff not $(\Gamma \models \bot)$.

Completeness Theorem

$(\forall \Gamma)$ Γ consistent implies Γ satisfiable

i.e.

not $\Gamma \cup \{\neg\varphi\} \vdash \bot$ implies not $\Gamma \cup \{\neg\varphi\} \models \bot$

i.e. <u>not</u> $\Gamma \vdash \varphi$ implies <u>not</u> $\Gamma \models \varphi$

i.e. $\Gamma \models \varphi$ implies $\Gamma \vdash \varphi$ (Completeness (objective)).

We alluded to $\Gamma \cup \{\neg\varphi\} \vdash \perp$ iff $\Gamma \vdash \varphi$

Suppose $\Gamma \cup \{\neg\varphi\} \vdash \perp$. then by RAA, $\Gamma \vdash \varphi$.

Conversely, if $\Gamma \vdash \varphi$ then $\Gamma \cup \{\neg\varphi\} \vdash \varphi \wedge \neg\varphi$.

So $\Gamma \cup \{\neg\varphi\} \vdash \perp$ from $\neg$ elimination.

Similary $\Gamma \cup \{\neg\varphi\} \models \perp$ iff $\Gamma \models \varphi$.

If $\Gamma \cup \{\neg\varphi\} \models \perp$, no v satisfies both Γ and $\neg\varphi$.

So any v satisfying Γ does not satisfy $\neg\varphi$,

i.e. satisfies $\neg\neg\varphi$.

Conversely, if $\Gamma \models \varphi$, then if v satisfies Γ

it <u>must</u> satisfy φ, i.e. it cannot satisfy $\neg\varphi$.

So no v satisfies $\Gamma \cup \{\neg\varphi\}$.

Proof that consistency implies satisfiability (sketch)

Any consistent set Γ can be extended

to one Γ' that is maximally consistent:

- Γ' is consistent
- Γ' is not a proper subset of any consistent set

by recursively adding formulas to Γ ("saturating" it).

A maximally consistent set must be

closed under derivability:

If $\Gamma' \vdash \varphi$ then $\varphi \in \Gamma'$.

Proof If $\Gamma' \vdash \varphi$ but $\varphi \notin \Gamma'$, then

$\Gamma' \cup \{\varphi\}$ is consistent. For if not

then $\Gamma' \cup \{\varphi\} \vdash \bot$. But since Γ' is

consistent, this could happen only if $\neg\varphi \in \Gamma'$;

but if $\neg\varphi \in \Gamma'$ and $\Gamma' \vdash \varphi$, $\Gamma' \vdash \bot$, contradiction.

Since $\Gamma' \subset \Gamma' \cup \{\varphi\}$, Γ' couldn't have been maximal.

If Γ' is maximally consistent, then

for any φ, either

$$\varphi \in \Gamma'$$

or $\neg\varphi \in \Gamma'$

but not both.

Furthermore, $\varphi \to \psi \in \Gamma'$

iff $\varphi \in \Gamma'$ implies $\psi \in \Gamma'$

For proofs of the above, see van Dalen.

<u>Every maximally consistent set Γ' is satisfiable</u>

Define a valuation v thus:

For any proposition symbol p,

$$v(p) = \begin{cases} 1 & \text{if } p \in \Gamma' \\ 0 & \text{otherwise} \end{cases}$$

v <u>induces</u> a truth value for each

formula in Γ'.

Claim For all $\varphi \in \Gamma'$, $v(\varphi) = 1$.

The proof is by structural induction on formulas.

Basis By definition, the claim is true for

proposition symbols in Γ',

Induction For formulas of form $\varphi \to \psi$

$v(\varphi \to \psi) = 1$ iff $(v(\varphi) = 0$ or $v(\psi = 1))$

iff $(\varphi \notin \Gamma'$ or $\psi \in \Gamma')$ by the induction hypothesis

iff $\varphi \to \psi \in \Gamma$ by closure under derivation

(If $\Gamma' \vdash \psi$ then $\Gamma' \vdash \varphi \to \psi$, by $\to I$.

If not $\Gamma' \vdash \varphi$, then $\Gamma' \vdash \neg\varphi$ by maximality,

so $\Gamma' \vdash \varphi \to \psi$ by $\neg E$ and $\to I$.)

For formulas of the form $\varphi \wedge \psi$

$v(\varphi \wedge \psi) = 1$ iff $v(\varphi) = v(\psi) = 1$

iff $\varphi \in \Gamma$ and $\psi \in \Gamma$ by the induction hyp.

iff $\Gamma \vdash \varphi \wedge \psi$ by closure under derivability.

by virtue of $\wedge I$ and $\wedge E$.

We showed how a maximal extension Γ' of a consistent set Γ is satisfiable. Hence Γ itself is satisfiable.

Thus every consistent set is satisfiable, which is the completeness theorem.

A constructive proof, for Γ finite.

The proof of completeness above showed us that a proof of $\Gamma \vdash \varphi$ exists whenever $\Gamma \models \varphi$, but did not say how to construct that proof.

We would like an alternate, constructive, proof that $\models \varphi$ implies $\vdash \varphi$.

Then for Γ finite $\{\varphi_1, \dots, \varphi_n\}$ we can appeal to $\Gamma \models \varphi$ iff $\Gamma \models \varphi_1 \to (\varphi_2 \to (\dots \to (\varphi_n \to \varphi)))$ and $\Gamma \vdash \varphi$ iff $\Gamma \vdash \varphi_1 \to (\varphi_2 \to \dots \to (\varphi_n \to \varphi))$.

Proof that $\models \varphi$ implies $\vdash \varphi$.

We use structural induction on φ.

The result follows from a generalization:

Let $p_1, \dots, p_n$ be the proposition symbols occuring in φ.

Let $p_1^*, \dots, p_n^*$ be either the proposition symbols or their negations, depending on the corresponding row of a <u>truth table</u> for φ.

$$p_i^* = \begin{cases} p_i & \text{if the stub entry is } 1 \\ \neg p_i & \text{if} \quad \text{"} \quad 0 \end{cases}$$

Similarly, given a row of the table, define

$$\varphi^* = \begin{cases} \varphi & \text{if the } \varphi \text{ entry is } 1 \\ \neg\varphi & \text{if} \quad \text{"} \quad 0 \end{cases}$$

syntactic

The generalization to be proved is:

$$\{p_1^*, p_2^*, \dots, p_n^*\} \vdash \varphi^*$$

for <u>each</u> row of the truth table.

In particular, if φ is a tautology, then

φ^* will always be φ. So by a series of LEM and $\vee$ Elimination steps, we will have a proof of φ from the individual proofs

$$\{p_1^*, \dots p_n^*\} \vdash \varphi^*$$

It is the existence of the individual proofs that is shown by <u>structural induction</u>:

<u>Basis</u> If φ is p_i then $\{p_i\} \vdash p_i$ trivially.

<u>Induction</u> Suppose φ is $\psi \wedge \xi$.

The induction hyp is that

$$\{p_1^*, \dots, p_n^*\} \vdash \psi^*$$

and

$$\{p_1^*, \dots, p_n^*\} \vdash \xi^*$$