Your name ______________________________

Harvey Mudd College
CS 81 Final Exam
Fall semester, 2006

Part 1 Closed Book

1. This exam is in two parts.
2. For part 1, no reference materials are to be used. Part 1 counts for 25%.
3. You should probably not work on part 1 more than 45 minutes.
4. After submitting your solutions for part 1, you may begin part 2, wherein you may consult a single 2-sided crib sheet. Part 2 counts for 75%.
5. Once you start part 2, it is not permitted to return to part 1.
6. The exam has a 3-hour overall time limit for both parts combined.
7. It is best to spend your time working on problems that pay off in terms of the most points.

1. [5 points]

Consider the following famlies of languages that we have studied:

- regular
- context-free
- recursive
- recursively-enumerable

In what important way is the family “recursive” different from all of the others?

2. [10 points]

For each language row below, check the *left-most* column entry for which the language is in that family. Use the row numbers should you wish to attach a note clarifying your answer to any item. (For the rows involving formulas, assume that the variables and proposition symbols in the formula are represented using a finite alphabet, e.g. proposition symbols are p, p', p'', p''', … so that a finite alphabet can be used overall.)

		Regular	Context-Free	Recursive	Recursively-Enumerable	None of the preceding
1	The set of all regular expressions.					
2	The set of all Turing machine encodings, using an encoding such as the one in our text.					
3	The set of all Turing machine encodings of Turing machines that always halt when started with their own description as input.					
4	The set of all surnames in the most recent Los Angeles telephone directory.					
5	The set of all predicate logic formulas.					
6	The set of all universally-valid predicate logic formulas.					
7	The set of all satisfiable predicate logic formulas.					
8	The set of all propositional logic formulas.					
9	The set of all satisfiable propositional logic formulas.					
10	The empty set.					

3. [5 points]

I.M. Brilliant has devised a new Turing-machine encoding that he thinks will "help solve a variant of the halting problem". Instead of using $M_0, M_1, M_2, \ldots$ as the enumeration of all Turing machines using the encoding discussed in the text, I.M. suggests encoding the Turing machines as $N_0, N_1, N_2, \ldots$ in the following way:

a. As sets, the two sets of machines are equal $\{M_0, M_1, M_2, \ldots\}=\{N_0, N_1, N_2, \ldots\}$, so both cover all of the machines.

b. The even-numbered $N_0, N_2, N_4, \ldots$ correspond to machines that halt on every input.

c. The odd-numbered machines in $N_1, N_3, N_5, \ldots$ correspond to machines that fail to halt on at least one input.

Comment on I.M.'s approach, specifically on its general utility and a possible algorithm for transforming the new numbering into the old numbering.

4. [5 points]

Goldbach's conjecture is that every even integer > 2 is the sum of two primes (not necessarily distinct). Consider the language over {0, 1}:

L = {x | x = 1 if Goldbach's conjecture is true, 0 otherwise}

Show that L is recursive.

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Part 2

1. For this part, you may use a single 2-sided crib sheet, but no other reference material.
2. Please turn in your crib sheet along with the exam. It will be returned to you.
3. Part 2 counts for 75%.
4. You are not permitted to return to part 1 after starting part 2.
5. The exam has a 3-hour overall time limit for both parts combined.
6. It is best to spend your time working on problems that pay off in terms of the most points.

5. [10 points]

The sequent below states that the composition of two 1-1 functions is itself 1-1.
Prove it using natural deduction.

$\forall x \forall y\ f(x) = f(y) \rightarrow x = y$,	(premise)
$\forall x \forall y\ g(x) = g(y) \rightarrow x = y$	(premise)
$\vdash \forall x \forall y\ f(g(x)) = f(g(y)) \rightarrow x = y$	(conclusion)

6. [15 points]

   Prove using natural deduction:

   $\vdash \exists x\, (\varphi \wedge \psi) \rightarrow ((\exists x \varphi) \wedge (\exists x \psi))$

7. [15 points]

Using the tree method, establish whether or not the following formula is valid.

$(\exists x\ (P(x) \lor Q(x))) \leftrightarrow ((\exists x\ P(x)) \lor (\exists x\ Q(x)))$

8. [20 points]

"Polish" notation is a notation for terms that doesn't require parentheses. It relies on each function symbol having a distinct arity. For simplicity, we consider here a term language with just two symbols, a 2-ary function symbol, **f**, and a constant symbol, **a**. In this case, the terms in Polish notation are given by:

a. The constant symbol **a** is a term.
b. If *X* and*Y* are terms, then so is **f***XY*.
c. The only terms are those derived by the above rules.

For example, here are five terms: **a**, **faa**, **fafaa**, **ffaafaa**, **ffaafafaa**.

Let L be the language of all terms as above.

a. [2/20] Give a context-free grammar for L.
b. [8/20] Diagram the abstract states of L and the transitions between them.
c. [4/20] Show that L is not regular.
d. [6/20]Give the transition rules for a pushdown automaton that accepts L.

9. [15 points]

The sequent below states that the composition of two onto functions is itself onto.

Prove it by the resolution method.

$\forall y \exists x\ f(x) = y$ (premise)

$\forall y \exists x\ g(x) = y$ (premise)

$\vdash \forall y \exists x\ f(g(x)) = y$ (conclusion)

Here is an equivalent clause form, to help you get started, where clauses 4-6 represent facts about equality that will likely be needed.

1. $E(f(i(y)), y)$.
2. $E(g(j(y)), y)$.
3. $\neg E(f(g(x)), c)$.
4. $\neg E(x, y) \lor \neg E(y, z) \lor E(x, z)$.
5. $\neg E(x, y) \lor E(f(x), f(y))$.
6. $\neg E(x, y) \lor E(g(x), g(y))$.

Here **E** represents the equality predicate, **i** is a Skolem function representing the inverse of **f**, **j** is a Skolem function representing the inverse of **g**, and **c** is a Skolem constant that arises from negating the conclusion. (There is more that can be said about **E**, but this is enough to enable the proof.)