1. Convert the following NFA to a DFA using the algorithm given in class.
2. Convert the NFA in problem 1 into a regular expression using the algorithm described in class. Show your work.
Eliminate old start state

Eliminate states next to start state
3. Prove the following *improved* pumping lemma. Let $L$ be any regular language. Then there exists an $n$ such that for any $x$ in $L$, for any way of writing $x$ as $z_1z_2z_3$ with $|z_2| \geq n$, there exists strings $w_1$, $w_2$, and $y$ such that $w_1yw_2 = z_2$, $|w_1y| \leq n$, $|y| > 0$, and the string $z_1w_1y^kw_2z_3$ is in $L$ for all $k \geq 0$. 

**Diagram:**

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Eliminate states next to start state

ab(ab)* + aa(ba)*
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Suppose L is regular. Then there is a DFA, $M=(Q,A,\delta,q_0,F)$ accepting L. Assume $M$ has $n$ states. Suppose $x=z_1z_2z_3$ is a string in L where $|z_2|\geq n$. Let $q_0=\delta^*(q_0,z_1)$ and let $q_1, q_2, \ldots, q_n$ be the states entered while reading the first $n$ symbols of $z_2$. Since $M$ only has $n$ state, the sequence $q_0,\ldots,q_n$ must include some state twice. Assume $q_i=q_{i+j}$. Let $w_1$ be the prefix of $z_2$ with length $i$. Let $y$ be the next $j$ symbols of $z_2$. And let $w_2$ be the remainder of $z_2$. Note that $y$ labels the loop through state $q_i=q_{i+j}$. This loop can be eliminated or repeated and the resulting path is still an accepting path. Therefore the string $z_1w_1yw_2z_3$ is in L for all $k\geq 0$.

4. Which of the following languages over $A=\{a,b\}$ are regular? Prove your answer. (Hint: You may find the improved pumping lemma particularly helpful for at least one of these problems.)

a. $\{a^{3n} \mid n\geq 1\}$
   This language is regular. It is generated by $aaa(aaa)^*$. 

b. $\{a^ib^ia^k \mid i>j>k\}$
   Suppose this language is regular. Consider the string $x=a^{n+1}b^na^{-1}$, where $n$ is provided by the pumping lemma. This string is in the language. By the pumping lemma we can write $x$ as $a^ia^{-i}a^jb^nba^{-j}$ for some $i+j\leq n$ and $j>0$ such that $a^ia^{-i}a^jb^nba^{-j}$ is in the language for all $k\geq 0$. When $k=0$, we get the string $x'=a^{n+1-j}b^na^{-j}$. Since $j>0$, $x'$ is not in the language, which is a contradiction. Therefore this language is not regular.

c. $\{a^ib^ia^k \mid i+j+k \text{ is even}\}$
   This language is regular. It is generated by $(aa)^*(bb)^*(aa)^* + a(aa)^*b(bb)^*(aa)^* + a(aa)^*(bb)^*a(aa)^* + (aa)^*b(bb)^*a(aa)^*$. 

d. $\{xwx^R \mid x,w \text{ are in } A^* \text{ and } x\neq\varepsilon\}$ (Note: $x^R$ denotes the string $x$ in reverse order)
   This language is regular. It is generated by $a(a+b)^*a + b(a+b)^*b$. 

e. $\{x \mid x=x^R\}$
   Suppose this language is regular. Consider the string $u=a^nb^na$, where $n$ is provided by the pumping lemma. This string is in the language. By the pumping lemma we can write $u$ as $a^ia^nb^na$ for some $i+j\leq n$ and $j>0$ such that $a^ia^nb^na$ is in the language for all $k\geq 0$. When $k=0$, we get the string $u'=a^{n-j}b^na$. Since $j>0$, $u'$ is not in the language, which is a contradiction. Therefore this language is not regular.

f. $\{xx^Rw \mid x,w \text{ are in } A^* \text{ and } x\neq\varepsilon\}$
   Suppose this language is regular. Consider the string $u=(ab)^n(ba)^n$, where $n$ is provided by the improved pumping lemma. This string is in the language. By the improved pumping lemma we can write $u$ as
Let $L$ be a regular language. Which of the following are also regular? Prove your answer.

a. $L^R = \{ x | x^R \in L \}$

This language is regular. Let $M=(Q,A,\delta,q_0,F)$ be a DFA accepting $L$. We’ll construct an NFA $M^R$ such that $L(M^R)=L^R$.

The states of $M^R$ are $Q \cup \{q_R\}$, where $q_R$ is a new state and also the start state of $M^R$. The only final state of $M^R$ is $q_S$; i.e. the final state in $M$. For every transition $\delta(q,a)=q'$ in $M$ there is a reverse transition $\delta_R(q',a)=q$ in $M^R$. In addition there are $\varepsilon$-transitions from the start state $q_R$ of $M^R$ to every $q$ in $M^R$ such that $q$ is a final state in $M$.

Claim: $L^R=L(M^R)$.

$\Rightarrow$ Suppose $x^R$ is in $L^R$. Then $x$ is in $L$ and there is an accepting path $q_Sq_1q_2\ldots q_m$ in $M$. By construction, $q_m\ldots q_2q_1q_S$ is a path in $M^R$ labeled $x^R$. Since $q_m$ must be a final state in $M$, there is an $\varepsilon$-transition in $M^R$ from $q_R$ to $q_m$. Since $q_S$ is a final state in $M^R$, $q_Rq_m\ldots q_2q_1q_S$ is an accepting path for $x^R$ in $M^R$.

$\Leftarrow$ Suppose $x^R$ is accepted by $M^R$. Then there is some accepting path $q_Rq_1q_2\ldots q_mq_S$. By the construction $q_Sq_m\ldots q_2q_1$ must be a path labeled $x$ in $M$. Since there is an edge from $q_R$ to $q_1$ in $M^R$, $q_1$ must be a final state in $M$. Since $q_S$ is the start state of $M$, this path accepts $x$. Thus $x$ is in $L$ and $x^R$ is in $L^R$.

b. $\frac{1}{2} L = \{ x | \exists y | |y|=|x| \text{ and } xy \in L \}$

This language is regular. Let $M=(Q,A,\delta,q_0,F)$ be a DFA recognizing $L$.

Let $q$ be any state in $M$. Consider the languages $L_{q,0} = \{ x | \delta^*(q_0,x)=q \}$ and $L_{q,1} = \{ x | \text{ for some } y, |y|=|x|, \delta^*(q,y) \in F \}$. Since regular languages are closed under finite union and finite intersection, it follows from Claims 1-3 that $\frac{1}{2} L$ is regular.

Claim 1: $L_{q,0}$ is regular.
Claim 2: $L_{q,1}$ is regular.

Claim 3: $\frac{1}{2} L = \bigcup_{q \in Q} (L_{q,0} \cap L_{q,1})$

Proof of Claim 1: This language is recognized by the DFA $(Q,A,\delta,q_0,\{q\})$.

Proof of Claim 2: Consider the NFA $M_{q,1} = \{Q,A,\chi,q,F\}$ where $\chi(q,a) = \{q_j \mid \exists b \in A, \delta(q,b) = q_j\}$. Note that $M$ is a DFA and has no $\epsilon$-transitions. The construction of $M_{q,1}$ did not introduce an $\epsilon$-transitions. So neither $M$ nor $M_{q,1}$ have any $\epsilon$-transitions.

It is easy to see that $q' \in \chi^*(q,x)$ iff there is a path of length $|x|$ from $q$ to $q'$ in $M$. (Note this would not hold if either machine had $\epsilon$-transitions.) We omit the obvious proof by induction on $|x|$.

It follows that $L_{q,1} = L(M_{q,1})$.

Proof of Claim 3: This is obvious from the definitions.

c. $\{x \mid xx^R \in L\}$

This language is regular. Let $M=(Q_M,A,\delta_M,q_s,F_M)$ be a DFA that recognizes $L$.

Let $q$ be any state in $M$ and consider the languages $L_{q,0} = \{x \mid \delta^*_M(q_s,x) = q\}$ and $L_{q,1} = \{x \mid \delta^*_M(q,x^R) \in F\}$. Since regular languages are closed under finite intersection and finite union, it follows from Claims 1-3 that this language is regular.

Claim 1: $L_{q,0}$ is regular.
Claim 2: $L_{q,1}$ is regular.

Claim 3: $\{x \mid xx^R \in L\} = \bigcup_{q \in Q} (L_{q,0} \cap L_{q,1})$

Proof of Claim 1: This language is recognized by $(Q_M,A,\delta_M,q_s,\{q\})$.

Proof of Claim 2: To see this, consider the reverse of the DFA $(Q_M,A,\delta_M,q,F_M)$; i.e. use the construction procedure described in the solution to 1a. This machine accepts strings that label paths from a final state in $M$ to the state $q$, but this is exactly $L_{q,1}$.

Proof of Claim 3:
Suppose $x$ is $\{x \mid xx^R \in L\}$. Then $xx^R \in L$. Assume $\delta_M^*(q_s,x) = q$. Then $x \in L_{q,0}$. Also, since $\delta_M^*(q_s,xx^R) \in F$, it must be the case that $\delta_M^*(q,x^R) \in F$. Therefore $x$ is in $L_{q,1}$. Thus $x$ is in $\cup_{q \in Q}(L_{q,0} \cap L_{q,1})$.

<=$x$ is $\cup_{q \in Q}(L_{q,0} \cap L_{q,1})$. Then for some state $q$ in $M$, $x \in \{x \mid \delta_M^*(q_s,x) = q\} \cap \{x \mid \delta_M^*(q,x^R) \in F\}$. Thus there is a path from $q_s$ to $q$ in $M$ with label $x$ and a path from $q$ to some final state in $M$ labeled $x^R$. The concatenation of these paths is an accepting path for $xx^R$. Thus $xx^R$ is in $L$.

d. $\{xz \mid \exists y \mid |y|=|x|=|z| \text{ and } xyz \in L\}$

This is not, in general, regular.

Proof: Let $L$ be the regular language over $\{a,b,#\}$ generated by $a^*#b^*$. Let $L' = \{xz \mid \exists y \mid |y|=|x|=|z| \text{ and } xyz \in L\}$. Notice that $L' = \{w \mid w \in L \text{ and } |w| \text{ is even}\} \cup \{a^ib^i\}$. If $L'$ is regular then so is $L'-L = \{a^ib^i \mid i>0\}$. But this set is clearly not regular, so neither is $L'$. 