

Natural Deduction Proofs

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Multiple Proof Systems Exist

with multiple variants for each system

- Hilbert system (several axioms, one rule)
- Gentzen system (no axioms, several rules)
 - Natural deduction
 - Sequent calculus
- Tableau systems
- Resolution (used for mechanization)

Natural Deduction

- Possibly closest to the way in which proofs are usually informally presented, yet
- Formal
- This approach can be used in your meta-proofs as well.

“Formal”

- “Formal” means the rules are strictly syntactic, based on symbol manipulation.
- No appeal to intuition or informal understanding is required (except perhaps when deciding which rule to apply).
- A computer can check whether the rules of a formal system are applied correctly.

Object Language

- We assume proposition symbols: p, q, r, s, \dots
- We assume connectives: $\wedge \vee \neg \rightarrow \leftrightarrow$
- We use special proposition symbols:
 - \perp (read “bottom”, “false”, “falsum”).
 - \top (read “top”, or “true”)
- Regarding assignments v discussed previously, **always**:
 - $v(\perp) = F$ (false)
 - $v(\top) = T$ (true)
- although the assignments will not matter until we discuss meaning.

Meta-Language

\vdash (turnstile) represents **provability**

Γ is a set of formulas (called **premises**)

F is a formula (called the **conclusion**)

$\Gamma \vdash A$

means that A is provable using formulas in Γ (and rules to be introduced).

A, B, C, \dots represent formulas in the meta language

p, q, r, \dots represent proposition symbols in the object language

The Two Turnstiles

\vdash (turnstile) represents **provability**

\vDash (double turnstile) represents **entailment**
(defined by truth-functional interpretations)

Ideally, $\Gamma \vdash A$ iff $\Gamma \vDash A$, but this is not automatic.

It depends on the proof rules.

Rule Notation

- Horizontal bar separates antecedents (0 or more) from consequent (one).

Natural Deduction Philosophy

- For each connective $\wedge \vee \neg \rightarrow \leftrightarrow$ there are simple rules for:
 - Introducing the connective
 - Eliminating the connective

\wedge -Introduction Rule

$$\frac{A \quad B}{A \wedge B} \quad \wedge I$$

Proof Formats

- Proofs can be presented as either
 - Trees
 - Tabulations (Fitch diagrams, box diagrams)

Tabulations are easier for beginners.

Tabulations are more verbose, but also avoid replications that are sometimes necessary with trees.

$$p, q \mid\text{---} q \wedge p$$

- Notation:
p, q abbreviates the set of formulas {p, q}.
q \wedge p is one formula
- This says: Formula “q \wedge p” is provable from the set of formulas {p, q}.
- Note: We don't need to use everything in the set.
- This utterance is called a **Sequent**.

Proof of: $p, q \mid\text{---} q \wedge p$

Using a tabulation

Line #	Formula	Justification
1.	p	premise
2.	q	premise
3.	$q \wedge p$	$\wedge I(2, 1)$

Proof of: $p, q \mid\text{---} q \wedge p$

Using a tree:

The leaves (at top) are premises.

The root is the conclusion.

$$\frac{\underline{q} \quad \underline{p}}{q \wedge p} \quad \wedge I$$

Prove: $p, q, r \mid\text{---} (q \wedge r) \wedge p$

- Using a tabulation
- Using a tree

\wedge -Elimination Rules

$$\frac{A \wedge B}{A}$$

A

$\wedge E(L)$

$$\frac{A \wedge B}{B}$$

B

$\wedge E(R)$

L means the left sub-formula is preserved.

R means the right sub-formula is preserved.

Proof of: $p \wedge q \mid\text{---} q \wedge p$

Using a tabulation

1.	$p \wedge q$	premise
2.	p	$\wedge E(L, 1)$
3.	q	$\wedge E(R, 1)$
4.	$q \wedge p$	$\wedge I(2, 1)$

Proof of: $p \wedge q \mid\text{---} q \wedge p$

Using a tree:

$$\begin{array}{ccc}
 \underline{p \wedge q} \wedge E(R) & & \underline{p \wedge q} \wedge E(L) \\
 \underline{q} \text{-----} \underline{p} & & \wedge I \\
 & & q \wedge p
 \end{array}$$

Note that the premise was replicated in the tree, but not in the tabulation.

In general, trees may require replication of larger sub-trees.

Vision

- These forms of proof obviously require more effort than a simple truth table, so why bother?
- These are about “reasoning”, and will ultimately be shown equivalent to truth-tables.
- More importantly, these **extend** to predicate calculus, where truth-tables certainly do not.

\vee -Introduction Rules

$$\frac{A}{A \vee B} \quad \wedge I(L)$$

$$\frac{B}{A \vee B} \quad \wedge I(R)$$

\vee -Introduction Purpose

- As \vee -Introduction **loses** information, its main use is to put the information in a form needed for the application of other rules.
- It is not usually seen as the last line of a proof.
- The same is true for \wedge -Elimination.

\forall -Elimination Rule

We defer this until after \rightarrow -Introduction.

\rightarrow -Elimination Rule

$$\frac{A \quad A \rightarrow B}{B} \rightarrow E$$

This is a very important rule.

It is also known as ***Modus Ponens*** (MP) (Latin: “mode that affirms by affirming”)

or **detachment**.

(This is the main rule in Hilbert-type systems.)

Proof of: $p \wedge q, p \rightarrow (q \rightarrow r) \mid \text{---} r$

1.	$p \wedge q$	premise
2.	p	$\wedge E(L, 1)$
3.	q	$\wedge E(R, 1)$
4.	$p \rightarrow (q \rightarrow r)$	premise
5.	$q \rightarrow r$	$\rightarrow E(2, 4)$
6.	r	$\rightarrow E(3, 5)$

\neg -Elimination Rule

$$\frac{A \quad \neg A}{\perp} \quad \neg E$$

In one view, $\neg A$ can be regarded as just an abbreviation for $A \rightarrow \perp$,

in which case $\neg E$ is just an instance of $\rightarrow E$.

Contradiction Rule

$$\frac{\perp}{A}$$

Contradiction

If bottom can be derived, anything can be derived.

This rule is mostly found in inner contexts, to be described.

Truth Rule

$\frac{\quad}{T}$ Truth

Truth can be introduced at any point.

This is not a very useful rule, because conveys no information. It is here for symmetry.

Rules with Sub-Derivations

- A sub-derivation makes one or more additional assumptions (or hypotheses).
- Derivation proceeds, using those assumptions, along with any formulas derived prior.
- Any formula derived in the sub-derivation is **conditioned** (by means of \rightarrow) on the assumptions.

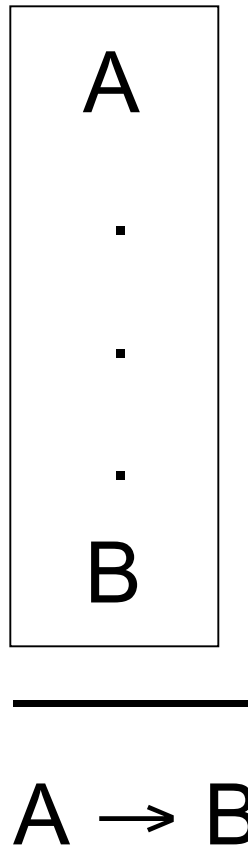
\rightarrow -Introduction Rule

- \rightarrow -Introduction is based on a **sub-proof**.
- Start with an additional assumption A.
- Derive B, using the assumption and any prior derived formulas.
- The sub-proof derives $A \rightarrow B$ outside.

- The assumptions themselves and formulas derived from them are **not usable outside** the sub-derivation.

- **Indentation**, or **boxing**, is used. No formula can be moved to the left of the indentation, or outside the box.

\rightarrow -Introduction Rule



$\rightarrow I$

Proof of: $p \rightarrow (q \rightarrow r) \mid \text{---} q \rightarrow (p \rightarrow r)$

1.	$p \rightarrow (q \rightarrow r)$	premise
2.	q	assumption
3.	p	assumption
4.	$q \rightarrow r$	$\rightarrow E(3, 1)$
5.	r	$\rightarrow E(2, 4)$
6.	$p \rightarrow r$	$\rightarrow I(3 \dots 5)$
7	$q \rightarrow (p \rightarrow r)$	$\rightarrow I(2 \dots 6)$

Suggestion for using $\rightarrow I$

- Work backward.
- Whenever the goal is of the form $A \rightarrow B$, consider using the $\rightarrow I$ rule, with a sub-derivation, of course.

Tree Format Sub-Proofs

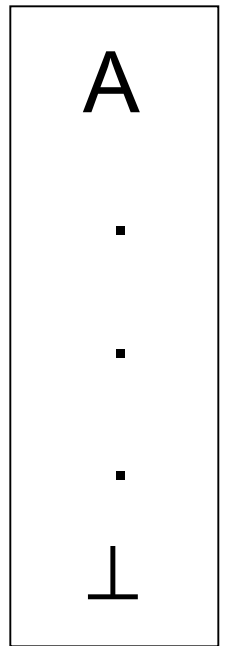
- The assumption is a leaf node of the tree.
- It is placed in **brackets** to denote that it is an assumption, not a premise, and a numeric reference number is attached.
- The assumption must be **discharged** (or “cancelled”) lower in the proof by an application of the $\rightarrow I$ rule, which carries the reference number of the assumption.

Proof of: $p \rightarrow (q \rightarrow r) \mid \text{---} q \rightarrow (p \rightarrow r)$

In tree form:

$$\begin{array}{c}
 \frac{\frac{\frac{[p]_1 \text{---} p \rightarrow (q \rightarrow r)}{[q]_2 \text{---} q \rightarrow r} \rightarrow E}{r} \rightarrow E}{p \rightarrow r} \rightarrow I_1}{q \rightarrow (p \rightarrow r)} \rightarrow I_2
 \end{array}$$

\neg -Introduction Rule



$\neg A$

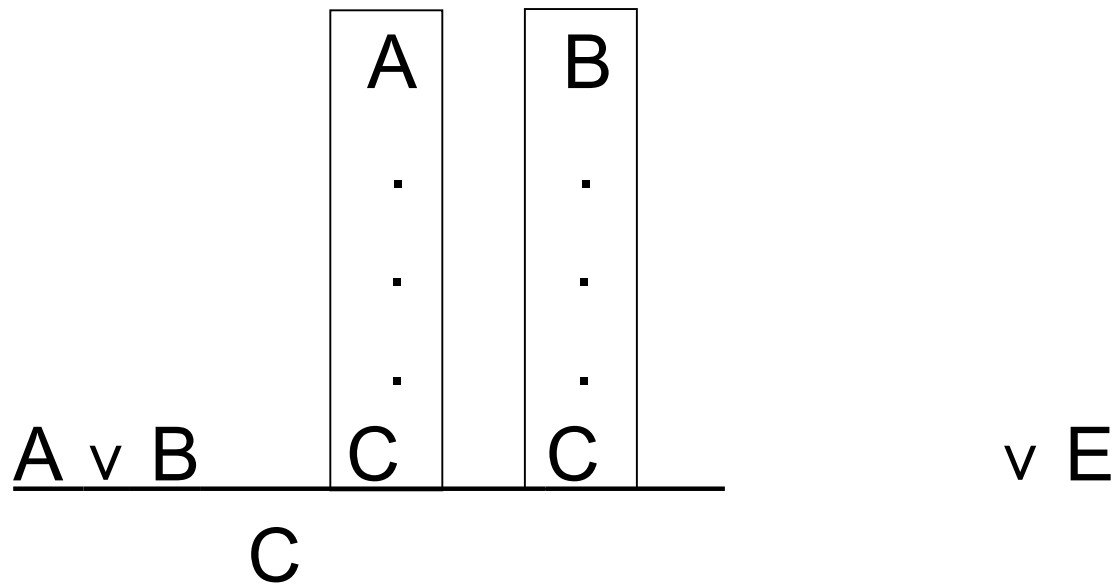
$\neg I$

\neg -Introduction Rule

- Analogous to \neg E, this rule can be seen as instance of \rightarrow I, under the viewpoint that $\neg A$ is an abbreviation for $A \rightarrow \perp$.
- This rule is one of two forms of “proof by contradiction”.
- Similar to \rightarrow I, a way to prove $\neg A$ is to work backward using this rule.

\vee -Elimination Rule

This rule uses *two* sub-derivations.



Guideline

- Try \vee -Elimination when a disjunction occurs in a working hypothesis.

Prove: $(p \vee q) \vee r \mid\text{---} p \vee (q \vee r)$

- A disjunction occurs in the premise.
- This suggests $\vee E$ be used.
- Each sub-proof starts with one of the disjuncts.
- Each sub-proof derives the same conclusion.
- $\vee E$ “glues” these proofs together to get an overall proof of the conclusion.

Prove: $(p \vee q) \vee r \vdash p \vee (q \vee r)$

- One sub-proof will be that for

$$r \vdash p \vee (q \vee r)$$

This will involve just two applications of \vee -introduction.

- The other sub-proof will be that for

$$(p \vee q) \vdash p \vee (q \vee r)$$

What will it involve?

Derived Rules

- Certain patterns tend to recur in proofs.
- These can be made into rules themselves.
- We are not adding to the basic rules, so much as creating abbreviations for these patterns.

Derived Rule Example

- We will have occasion to produce proofs that look like:

$A \rightarrow B$

$\neg B$

A
B
\perp

Assumption

$\rightarrow E$

$\neg E$

$\neg A$

$\neg I$

Derived Rule Example

- Rather than repeat this pattern, we introduce a rule that captures it:

$$\frac{A \rightarrow B \quad \neg B}{\neg A}$$

This rule is called **Modus Tollens** (MT)
(mode that denies by denying”).

We can use it, knowing that we can expand it to the other, more basic, rules if desired.

More Derived Rules

$$\frac{A \vee B \quad \neg A}{B}$$

This rule is called **Modus Tollendo Ponens** (MTP)
("mode that affirms by denying").

More Derived Rules

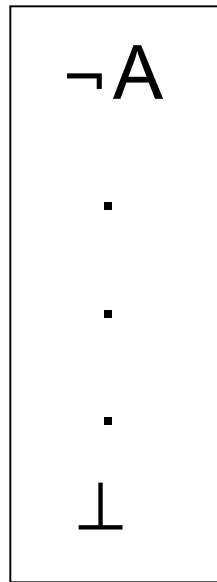
$$\frac{\neg(A \wedge B) \quad A}{\neg B}$$

This rule is called **Modus Ponendo Tollens** (MPT)
("mode that denies by affirming").

Classical vs. Constructive Logic

- Up until now, all rules are regarded as “constructive” (or “intuitionistic”) rules.
- The next one is not accepted in certain forms of logic, so is called **classical** rather than constructive, because it was from an earlier period before constructivity was appreciated.

RAA: NOT the \neg -Introduction Rule



A

RAA (Reductio ad absurdum)

RAA (Reductio ad absurdum)

- RAA looks deceptively similar to \neg Introduction.
- Since syntax is important, they are obviously not the same. Further, RAA can't be derived from the other rules.
- Technically, one derives a formula that has one more \neg . The other derives a formula with one fewer \neg .

Constructivism

- The constructivist school of mathematics places extra requirements on the form of proof.
- It does not regard a proof that $\neg A$ leads to a contradiction as a proof of A . Rather it is just a proof of $\neg\neg A$, which is not syntactically the same.
- There are greater implications for predicate logic.

Derived Rules Equivalent to RAA

- LEM (Law of the Excluded Middle):

$$\frac{}{A \vee \neg A} \quad \text{(no antecedent)} \\ \text{LEM}$$

- Double Negation Elimination:

$$\frac{\neg \neg A}{A} \quad \text{DNE}$$

Constructive Requirements

- A constructive proof of $A \vee B$ requires a proof of either A or one of B .
- Thus, $A \vee \neg A$ is not regarded as a given. One would need to show one or the other.

A Non-Constructive Proof

- There exist irrational numbers a, b such that a^b is rational.
- Proof: Let $b = \sqrt{2}$, which can be shown irrational separately. By the LEM, either b^b is rational or it isn't.
- If b^b is rational, choose $a = b$ and we are done.
- If b^b is irrational, choose $a = b^b$. Then $a^b = (b^b)^b = b^{bb} = b^2 = 2$, which is rational.
- After the proof, we still don't know what a is.

Basic Rules for Classical ND

- Contradiction, Truth
- $\wedge I$, $\wedge E$
- $\vee I$, $\vee E$
- $\rightarrow I$, $\rightarrow E$
- $\neg I$, $\neg E$
(which can be viewed as instances of $\rightarrow I$, $\rightarrow E$)
- RAA
(not intuitionistic, alternate equivalents: LEM, DNE)
- Gentzen called the rule sets NJ (intuitionistic) and NK (classical). He used DNE rather than LEM as basic.