Natural Deduction Proofs

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Multiple Proof Systems Exist

with multiple variants for each system

- Hilbert system (several axioms, one rule)
- Gentzen system (no axioms, several rules)
  - Natural deduction
  - Sequent calculus
- Tableau systems
- Resolution (used for mechanization)

Natural Deduction

- Possibly closest to the way in which proofs are usually informally presented, yet
- Formal

- This approach can be used in your meta-proofs as well.

“Formal”

- “Formal” means the rules are strictly syntactic, based on symbol manipulation.
- No appeal to intuition or informal understanding is required (except perhaps when deciding which rule to apply).
- A computer can check whether the rules of a formal system are applied correctly.

Object Language

- We assume proposition symbols: p, q, r, s, …
- We assume connectives: $\wedge \vee \neg \rightarrow \leftrightarrow$
- We use special proposition symbols:

  $\perp$ (read "bottom", "false", "falsum").

  $\top$ (read "top", or "true")
- Regarding assignments ν discussed previously, **always**:

  $\nu(\perp)$ = F (false)

  $\nu(\top)$ = T (true)
- although the assignments will not matter until we discuss meaning.

Meta-Language

$\vdash$ (turnstile) represents **provability**

Γ Is a set of formulas (called **premises**)
F is a formula (called the **conclusion**)

$\Gamma \vdash A$

means that A is provable using formulas in Γ (and rules to be introduced).

A, B, C, … represent formulas in the meta language

p, q, r, … represent proposition symbols in the object language

The Two Turnstiles

$\vdash$ (turnstile) represents **provability**

$\models$ (double turnstile) represents **entailment**
(defined by truth-functional interpretations)

Ideally, $\Gamma \vdash A$ iff $\Gamma \models A$, but this is not automatic.

It depends on the proof rules.

Rule Notation

- Horizontal bar separates antecedents (0 or more) from consequent (one).

Natural Deduction Philosophy

- For each connective $\wedge \vee \neg \rightarrow \leftrightarrow$ there are simple rules for:
  - Introducing the connective
  - Eliminating the connective

∧-Introduction Rule

$$\frac{A \qquad\qquad B}{A \wedge B}\ \wedge I$$

Proof Formats

- Proofs can be presented as either
  - Trees
  - Tabulations (Fitch diagrams, box diagrams)

Tabulations are easier for beginners.

Tabulations are more verbose, but also avoid replications that are sometimes necessary with trees.

$$p, q \vdash q \wedge p$$

- Notation:

  p, q abbreviates the set of formulas {p, q}.

  $q \wedge p$ is one formula

- This says: Formula "$q \wedge p$" is provable from the set of formulas {p, q}.

- Note: We don't need to use everything in the set.

- This utterance is called a **Sequent**.

Proof of: p, q $\vdash$ $q \wedge p$

Using a tabulation

Line #	Formula	Justification
1.	p	premise
2.	q	premise
3.	$q \wedge p$	$\wedge$I(2, 1)

Proof of: p, q $\vdash$ q ∧ p

Using a tree:

The leaves (at top) are premises.

The root is the conclusion.

$$\dfrac{q \qquad\qquad\qquad p}{q \wedge p}\ \wedge I$$

Prove: p, q, r $\vdash$ $(q \wedge r) \wedge p$

- Using a tabulation
- Using a tree

$\wedge$-Elimination Rules

$$\frac{A \wedge B}{A} \qquad \wedge E(L)$$

$$\frac{A \wedge B}{B} \qquad \wedge E(R)$$

L means the left sub-formula is preserved.

R means the right sub-formula is preserved.

Proof of: $p \wedge q \vdash q \wedge p$

Using a tabulation

1.	$p \wedge q$	premise
2.	p	$\wedge$E(L,1)
3.	q	$\wedge$E(R,1)
4.	$q \wedge p$	$\wedge$I(2, 1)

Proof of: $p \wedge q \vdash q \wedge p$

Using a tree:

$$\dfrac{\dfrac{p \wedge q}{q}\ {\scriptstyle \wedge E(R)} \qquad\qquad \dfrac{p \wedge q}{p}\ {\scriptstyle \wedge E(L)}}{q \wedge p}\ {\scriptstyle \wedge I}$$

Note that the premise was replicated in the tree, but not in the tabulation.

In general, trees may require replication of larger sub-trees.

Vision

- These forms of proof obviously require more effort than a simple truth table, so why bother?
- These are about “reasoning”, and will ultimately be shown equivalent to truth-tables.
- More importantly, these **extend** to predicate calculus, where truth-tables certainly do not.

∨-Introduction Rules

$$\frac{A}{A \vee B} \qquad \wedge I(L)$$

$$\frac{B}{A \vee B} \qquad \wedge I(R)$$

∨-Introduction Purpose

- As ∨-Introduction ***loses*** information, its main use is to put the information in a form needed for the application of other rules.
- It is not usually seen as the last line of a proof.
- The same is true for ∧-Elimination.

$\vee$-Elimination Rule

We defer this until after $\rightarrow$-Introduction.

$\rightarrow$-Elimination Rule

$$\frac{A \qquad A \rightarrow B}{B} \quad \rightarrow E$$

This is a very important rule.

It is also known as ***Modus Ponens*** (MP) (Latin: "mode that affirms by affirming")

or **detachment**.

(This is the main rule in Hilbert-type systems.)

Proof of: $p \wedge q,\ p \to (q \to r) \vdash r$

1.	$p \wedge q$	premise
2.	p	$\wedge E(L,1)$
3.	q	$\wedge E(R,1)$
4.	$p \to (q \to r)$	premise
5.	$q \to r$	$\to E(2, 4)$
6.	r	$\to E(3, 5)$

$\neg$-Elimination Rule

$$\frac{A \qquad \neg A}{\perp} \; \neg E$$

In one view, $\neg A$ can be regarded as just an abbreviation for $A \rightarrow \perp$,

in which case $\neg E$ is just an instance of of $\rightarrow E$.

Contradiction Rule

$$\frac{\perp}{A} \qquad \text{Contradiction}$$

If bottom can be derived, anything can be derived.

This rule is mostly found in inner contexts, to be described.

Truth Rule

$$\frac{\quad}{T} \qquad \text{Truth}$$

Truth can be introduced at any point.

This is not a very useful rule, because conveys no information. It is here for symmetry.

Rules with Sub-Derivations

- A sub-derivation makes one or more additional assumptions (or hypotheses).
- Derivation proceeds, using those assumptions, along with any formulas derived prior.
- Any formula derived in the sub-derivation is **conditioned** (by means of $\rightarrow$) on the assumptions.

$\rightarrow$-Introduction Rule

- $\rightarrow$-Introduction is based on a **sub-proof.**
- Start with an additional assumption A.
- Derive B, using the assumption and any prior derived formulas.
- The sub-proof derives $A \rightarrow B$ outside.

- The assumptions themselves and formulas derived from them are **not usable outside** the sub-derivation.

- **Indentation**, or **boxing**, is used. No formula can be moved to the left of the indentation, or outside the box.

→-Introduction Rule

$$\frac{\boxed{\begin{array}{c} A \\ \cdot \\ \cdot \\ \cdot \\ B \end{array}}}{A \rightarrow B} \quad \rightarrow I$$

Proof of: $p \rightarrow (q \rightarrow r) \vdash q \rightarrow (p \rightarrow r)$

1.	$p \rightarrow (q \rightarrow r)$	premise
2.	q	assumption
3.	p	assumption
4.	$q \rightarrow r$	$\rightarrow$E(3, 1)
5.	r	$\rightarrow$E(2, 4)
6.	$p \rightarrow r$	$\rightarrow$I(3…5)
7	$q \rightarrow (p \rightarrow r)$	$\rightarrow$I(2…6)

Suggestion for using $\rightarrow$I

- Work backward.
- Whenever the goal is of the form A$\rightarrow$B, consider using the $\rightarrow$I rule, with a sub-derivation, of course.

Tree Format Sub-Proofs

- The assumption is a leaf node of the tree.
- It is placed in **brackets** to denote that it is an assumption, not a premise, and a numeric reference number is attached.
- The assumption must be **discharged** (or “cancelled”) lower in the proof by an application of the $\rightarrow$I rule, which carries the reference number of the assumption.

Proof of: $p \to (q \to r) \vdash q \to (p \to r)$

In tree form:

$$\dfrac{\dfrac{\dfrac{[q]_2 \qquad \dfrac{[p]_1 \qquad p \to (q \to r)}{q \to r}\ {\to}E}{r}\ {\to}E}{p \to r}\ {\to}I_1}{q \to (p \to r)}\ {\to}I_2$$

¬-Introduction Rule

$$\frac{\boxed{\begin{array}{c} A \\ \cdot \\ \cdot \\ \cdot \\ \perp \end{array}}}{\neg A} \quad \neg I$$

¬-Introduction Rule

- Analogous to ¬E, this rule can be seen as instance of →I, under the viewpoint that ¬A is an abbreviation for $A \rightarrow \perp$.
- This rule is one of two forms of "proof by contradiction".
- Similar to →I, a way to prove ¬A is to work backward using this rule.

∨-Elimination Rule

This rule uses *two* sub-derivations.

$$\frac{A \vee B \qquad \boxed{\begin{matrix} A \\ . \\ . \\ . \\ C \end{matrix}} \qquad \boxed{\begin{matrix} B \\ . \\ . \\ . \\ C \end{matrix}}}{C} \qquad \vee E$$

Guideline

- Try $\vee$-Elimination when a disjunction occurs in a working hypothesis.

Prove: $(p \vee q) \vee r \vdash p \vee (q \vee r)$

- A disjunction occurs in the premise.
- This suggests $\vee$E be used.
- Each sub-proof starts with one of the disjuncts.
- Each sub-proof derives the same conclusion.
- $\vee$E "glues" these proofs together to get an overall proof of the conclusion.

Prove: $(p \vee q) \vee r \vdash p \vee (q \vee r)$

- One sub-proof will be that for

  $$r \vdash p \vee (q \vee r)$$

  This will involve just two applications of $\vee$-introduction.

- The other sub-proof will be that for

  $$(p \vee q) \vdash p \vee (q \vee r)$$

  What will it involve?

Derived Rules

- Certain patterns tend to recur in proofs.
- These can be made into rules themselves.
- We are not adding to the basic rules, so much as creating abbreviations for these patterns.

Derived Rule Example

- We will have occasion to produce proofs that look like:

$A \rightarrow B$	
$\neg B$	
A	Assumption
B	$\rightarrow$E
$\bot$	$\neg$E
$\neg A$	$\neg$I

Derived Rule Example

- Rather than repeat this pattern, we introduce a rule that captures it:

$$\frac{A \rightarrow B \qquad \neg B}{\neg A}$$

This rule is called **Modus Tollens** (MT) (mode that denies by denying”).

We can use it, knowing that we can expand it to the other, more basic, rules if desired.

More Derived Rules

$$\frac{A \lor B \qquad \neg A}{B}$$

This rule is called **Modus Tollendo Ponens** (MTP) (“mode that affirms by denying”).

More Derived Rules

$$\frac{\neg(A \wedge B) \qquad\qquad A}{\neg B}$$

This rule is called **Modus Ponendo Tollens** (MPT) ("mode that denies by affirming").

Classical vs. Constructive Logic

- Up until now, all rules are regarded as “constructive” (or “intuitionistic”) rules.
- The next one is not accepted in certain forms of logic, so is called **classical** rather than constructive, because it was from an earlier period before constructivity was appreciated.

RAA: NOT the ¬-Introduction Rule

$$\frac{\boxed{\begin{array}{c} \neg A \\ . \\ . \\ . \\ \bot \end{array}}}{A} \quad \text{RAA (Reductio ad absurdum)}$$

RAA (Reductio ad absurdum)

- RAA looks deceptively similar to ¬ Introduction.
- Since syntax is important, they are obviously not the same. Further, RAA can't be derived from the other rules.
- Technically, one derives a formula that has one more ¬. The other derives a formula with one fewer ¬.

Constructivism

- The constructivist school of mathematics places extra requirements on the form of proof.

- It does not regard a proof that $\neg A$ leads to a contradiction as a proof of A. Rather it is just a proof of $\neg\neg A$, which is not syntactically the same.

- There are greater implications for predicate logic.

Derived Rules Equivalent to RAA

- LEM (Law of the Excluded Middle):

$$\frac{\qquad}{A \vee \neg A} \quad \text{(no antecedent)} \quad \text{LEM}$$

- Double Negation Elimination:

$$\frac{\neg\neg A}{A} \quad \text{DNE}$$

Constructive Requirements

- A constructive proof of $A \vee B$ requires a proof of either A or one of B.

- Thus, $A \vee \neg A$ is not regarded as a given. One would need to show one or the other.

A Non-Constructive Proof

- There exist irrational numbers a, b such that $a^{\mathbf{b}}$ is rational.
- Proof: Let b = $\sqrt{2}$, which can be shown irrational separately. By the LEM, either $b^{\mathbf{b}}$ is rational or it isn't.
- If $b^{\mathbf{b}}$ is rational, choose a = b and we are done.
- If $b^{\mathbf{b}}$ is irrational, choose a = $b^{\mathbf{b}}$.Then $a^{\mathbf{b}}$ = $(b^{\mathbf{b}})^{\mathbf{b}} = b^{\mathbf{bb}} = b^{\mathbf{2}} = 2$, which is rational.
- After the proof, we still don't know what a is.

Basic Rules for Classical ND

- Contradiction, Truth
- $\wedge$I, $\wedge$E
- $\vee$I, $\vee$E
- $\rightarrow$I, $\rightarrow$E
- $\neg$I, $\neg$E
  (which can be viewed as instances of $\rightarrow$I, $\rightarrow$E)
- RAA
  (not intuitionistic, alternate equivalents: LEM, DNE)
- Gentzen called the rule sets NJ (intuitionistic) and NK (classical). He used DNE rather than LEM as basic.