



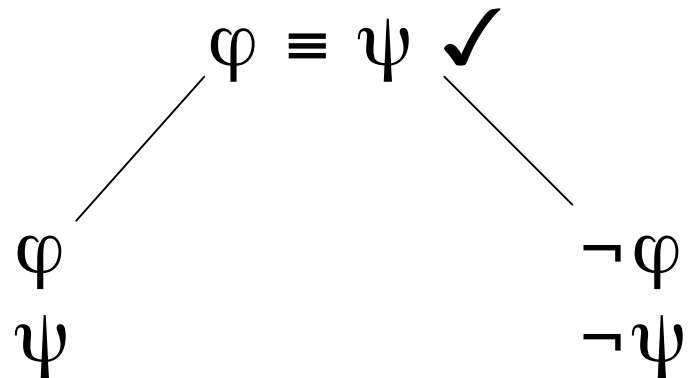
# Tableau Method for Predicate Logic

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February 2009



## Aside: Additional Tree Rules: $\varphi \equiv \psi$

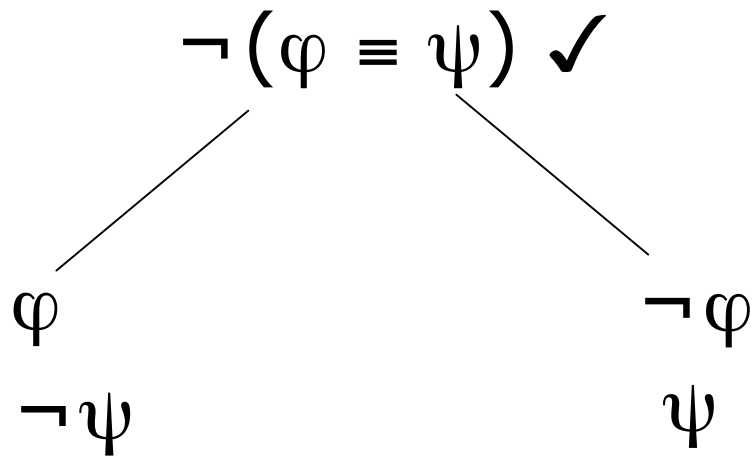
- This formula is retired and the tree **branches** with the formulas and their negations stacked:





## Additional Tree Rules: $\neg(\varphi \equiv \psi)$

- This formula is retired and the tree **branches**, with  $\varphi$  negated:





# Propositional Tree Rule Summary

stack	$\varphi \wedge \psi$	$\neg(\varphi \vee \psi)$	$\neg(\varphi \rightarrow \psi)$		$\neg \neg \varphi$
	$\varphi$ $\psi$	$\neg \varphi$ $\neg \psi$	$\varphi$ $\neg \psi$		$\varphi$
split	$\varphi \vee \psi$	$\neg(\varphi \wedge \psi)$	$\varphi \rightarrow \psi$	$(\varphi \equiv \psi)$	
	$\varphi$ $\psi$	$\neg \varphi$ $\neg \psi$	$\neg \varphi$ $\psi$	$\varphi$ $\neg \varphi$ $\psi$ $\neg \psi$	
				$\neg(\varphi \equiv \psi)$	
				$\varphi$ $\neg \varphi$ $\neg \psi$ $\psi$	



## $\neg\exists$ rule

$$\neg(\exists v) \varphi \quad \checkmark$$

$$(\forall v) \neg\varphi$$



## $\neg \forall$ rule

$$\neg (\forall v) \varphi \quad \checkmark$$

$$(\exists v) \neg \varphi$$



## $\exists$ rule

$$\frac{(\exists v) \varphi \quad \varphi[c/v]}{\varphi \checkmark}$$

where  $c$  is a **new** constant not appearing in the tree.

The rule can be used only once per  $\exists$  formula.



## $\forall$ rule

$(\forall v) \varphi$  Does not get a check!!  
 $\varphi[\tau/v]$

where  $\tau$  is any term free to replace  $v$  in  $\varphi$ .

This rule can be used arbitrarily-many times for a  $\forall$  formula.



# Example

$$\neg((\forall x)p(x) \rightarrow (\exists x)p(x))$$



# Example

$$\neg((\forall x)p(x) \rightarrow (\exists x)p(x)) \quad \checkmark$$

$$(\forall x)p(x)$$

$$\neg(\exists x)p(x)$$

# Example

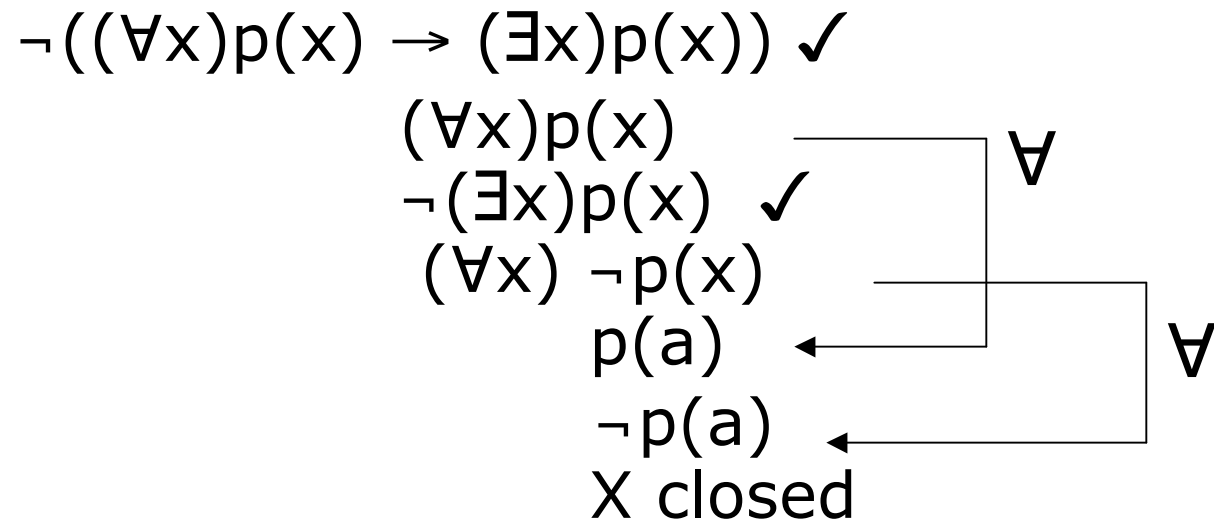
$$\neg((\forall x)p(x) \rightarrow (\exists x)p(x)) \quad \checkmark$$

$$(\forall x)p(x)$$

$$\neg(\exists x)p(x) \quad \checkmark$$

$$(\forall x) \neg p(x)$$

# Example



The root formula is not satisfiable.

$(\forall x)p(x) \rightarrow (\exists x)p(x)$  is valid

Closure depended on appropriate choice of term to substitute for  $x$  in  $(\forall x) \neg p(x)$ .



# Example

(in which  $\forall$  rule is used twice on the same line)

$$\neg(\exists x) (p(x) \rightarrow (\forall y)p(y))$$



# Example

$$\neg(\exists x) (p(x) \rightarrow (\forall y)p(y)) \checkmark$$
$$(\forall x) \neg(p(x) \rightarrow (\forall y)p(y))$$



# Example

$$\neg(\exists x) (p(x) \rightarrow (\forall y)p(y)) \checkmark$$

$$(\forall x) \neg(p(x) \rightarrow (\forall y)p(y))$$

$$\neg(p(a) \rightarrow (\forall y)p(y))$$



# Example

$$\neg(\exists x) (p(x) \rightarrow (\forall y)p(y)) \checkmark$$

$$(\forall x) \neg(p(x) \rightarrow (\forall y)p(y))$$

$$\neg(p(a) \rightarrow (\forall y)p(y)) \checkmark$$

$$p(a)$$

$$\neg(\forall y)p(y)$$

# Example

$$\neg(\exists x) (p(x) \rightarrow (\forall y)p(y)) \quad \checkmark$$

$$(\forall x) \neg(p(x) \rightarrow (\forall y)p(y))$$

$$\neg(p(a) \rightarrow (\forall y)p(y)) \quad \checkmark$$

$$p(a)$$

$$\neg(\forall y)p(y) \quad \checkmark$$

$$(\exists y) \neg p(y)$$

# Example

$$\neg(\exists x) (p(x) \rightarrow (\forall y)p(y)) \quad \checkmark$$

$$(\forall x) \neg(p(x) \rightarrow (\forall y)p(y))$$

$$\neg(p(a) \rightarrow (\forall y)p(y)) \quad \checkmark$$

$$p(a)$$

$$\neg(\forall y)p(y) \quad \checkmark$$

$$(\exists y) \neg p(y) \quad \checkmark$$

$$\neg p(b)$$

# Example

$$\neg(\exists x) (p(x) \rightarrow (\forall y)p(y)) \quad \checkmark$$

$$(\forall x) \neg(p(x) \rightarrow (\forall y)p(y))$$

$$\neg(p(a) \rightarrow (\forall y)p(y)) \quad \checkmark$$

$$p(a)$$

$$\neg(\forall y)p(y) \quad \checkmark$$

$$(\exists y) \neg p(y) \quad \checkmark$$

$$\neg p(b)$$

$$\neg(p(b) \rightarrow (\forall y)p(y)) \quad \leftarrow$$

# Example

$$\neg(\exists x) (p(x) \rightarrow (\forall y)p(y)) \quad \checkmark$$

$$(\forall x) \neg(p(x) \rightarrow (\forall y)p(y))$$

$$\neg(p(a) \rightarrow (\forall y)p(y)) \quad \checkmark$$

$$p(a)$$

$$\neg(\forall y)p(y) \quad \checkmark$$

$$(\exists y) \neg p(y) \quad \checkmark$$

$$\neg p(b)$$

$$\neg(p(b) \rightarrow (\forall y)p(y)) \quad \checkmark$$

$$p(b)$$

$$\neg(\forall y)p(y)$$

X

closes



# Exercise

- Show that  $(\exists x) (p(x) \rightarrow (\forall y)p(y))$  is valid (from basic definitions).
- Somehow, this looks wrong:

In a non-empty crowd,  
there is a person  $x$  such that  
(if  $x$  is a genius, everyone is a genius.)

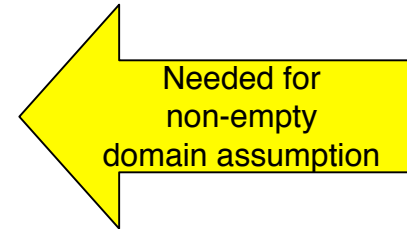
What's the catch?

- Give a natural deduction proof of it.  
(Hint: Use the LEM with  $(\forall y)p(y)$ .)



# Proof of $(\exists x) (p(x) \rightarrow (\forall y)p(y))$

1: actual i	premise
2: $\forall y.P(y) \vee \neg \forall y.P(y)$	Theorem $E \vee \neg E$
3: $\forall y.P(y)$	assumption
4: $P(i) \rightarrow \forall y.P(y)$	Theorem $E \vdash F \rightarrow E$ 3
5: $\exists x.(P(x) \rightarrow \forall y.P(y))$	$\exists$ intro 4,1
6: $\neg \forall y.P(y)$	assumption
7: $\exists z. \neg P(z)$	Theorem $\neg \forall x.R(x) \vdash \exists y. \neg R(y)$ 6
8: actual i1, $\neg P(i1)$	assumptions
9: $P(i1) \rightarrow \forall y.P(y)$	Theorem $\neg E \vdash E \rightarrow F$ 8.2
10: $\exists x.(P(x) \rightarrow \forall y.P(y))$	$\exists$ intro 9,8.1
11: $\exists x.(P(x) \rightarrow \forall y.P(y))$	$\exists$ elim 7,8-10
12: $\exists x.(P(x) \rightarrow \forall y.P(y))$	$\vee$ elim 2,3-5,6-11





# Termination

- Unlike the propositional case, the predicate version of tableaux does not necessarily terminate. This is because the  $\forall$  rule can be used arbitrarily-many times.
- It can be shown, however, that **if** the root formula is unsatisfiable, then **there exists** a closed tree for it.  
(If the root formula is satisfiable, the construction *might* not terminate.)



# Example of Non-Termination

$$(\forall x) (\exists y) (A(x) \wedge B(y))$$



# Example of Non-Termination

$$\begin{aligned} &(\forall x) (\exists y) (A(x) \wedge B(y)) \\ &(\exists y) (A(a) \wedge B(y)) \end{aligned}$$



# Example of Non-Termination

$(\forall x) (\exists y) (A(x) \wedge B(y))$

$(\exists y) (A(a) \wedge B(y)) \checkmark$

$(A(a) \wedge B(b_0))$



# Example of Non-Termination

$(\forall x) (\exists y) (A(x) \wedge B(y))$

$(\exists y) (A(a) \wedge B(y)) \checkmark$

$(A(a) \wedge B(b_0)) \checkmark$

$A(a)$

$B(b_0)$



# Example of Non-Termination

$(\forall x) (\exists y) (A(x) \wedge B(y))$

$(\exists y) (A(a) \wedge B(y)) \checkmark$

$(A(a) \wedge B(b_0)) \checkmark$

$A(a)$

$B(b_0)$

$(\exists y) (A(b_0) \wedge B(y))$



# Example of Non-Termination

$(\forall x) (\exists y) (A(x) \wedge B(y))$

$(\exists y) (A(a) \wedge B(y)) \checkmark$

$(A(a) \wedge B(b_0)) \checkmark$

$A(a)$

$B(b_0)$

$(\exists y) (A(b_0) \wedge B(y)) \checkmark$

$A(b_0) \wedge B(b_1)$



# Example of Non-Termination

$(\forall x) (\exists y) (A(x) \wedge B(y))$

$(\exists y) (A(a) \wedge B(y)) \checkmark$

$(A(a) \wedge B(b_0)) \checkmark$

$A(a)$

$B(b_0)$

$(\exists y) (A(b_0) \wedge B(y)) \checkmark$

$A(b_0) \wedge B(b_1) \checkmark$

$A(b_0)$

$B(b_1)$

# Example of Non-Termination

$$\begin{aligned} & (\forall x) (\exists y) (A(x) \wedge B(y)) \\ & \quad (\exists y) (A(a) \wedge B(y)) \checkmark \\ & \quad \quad (A(a) \wedge B(b_0)) \checkmark \\ & \quad \quad \quad A(a) \\ & \quad \quad \quad B(b_0) \\ & \quad (\exists y) (A(b_0) \wedge B(y)) \checkmark \\ & \quad \quad A(b_0) \wedge B(b_1) \checkmark \\ & \quad \quad \quad A(b_0) \\ & \quad \quad \quad B(b_1) \\ & \quad (\exists y) (A(b_1) \wedge B(y)) \\ & \quad \quad \dots \end{aligned}$$



# Example of Non-Termination

$$(\forall x) (\exists y) (A(x) \wedge B(y))$$
$$(\exists y) (A(a) \wedge B(y)) \checkmark$$
$$(A(a) \wedge B(b_0)) \checkmark$$
$$A(a)$$
$$B(b_0)$$

We can see at this point that the formula is satisfiable.

A 1-element interpretation will do, with domain  $\{0\}$ ,  
 $\mu(a) = \mu(b_0) = 0$ ,  $A = B = \{(0)\}$ .

# Using the Tableau Method to Find a Model in the Predicate Calculus

- $(\forall x) (A(x) \vee B(x)) \rightarrow ((\forall x)A(x) \vee (\forall x)B(x))$
- This formula is not valid, so its negation should be satisfiable.
  - $\neg((\forall x) (A(x) \vee B(x)) \rightarrow ((\forall x)A(x) \vee (\forall x)B(x))) \checkmark$
  - $(\forall x) (A(x) \vee B(x))$
  - $\neg((\forall x)A(x) \vee (\forall x)B(x)) \checkmark$
  - $\neg(\forall x)A(x) \checkmark$
  - $\neg(\forall x)B(x) \checkmark$
  - $(\exists x) \neg A(x)$
  - $(\exists x) \neg B(x)$



# Using the Tree Method to Find a Model in the Predicate Calculus

- So far we have unchecked:

$$(\forall x) (A(x) \vee B(x))$$

$$(\exists x) \neg A(x)$$

$$(\exists x) \neg B(x)$$



# Using the Tree Method to Find a Model in the Predicate Calculus

$(\forall x) (A(x) \vee B(x))$

$(\exists x) \neg A(x) \checkmark$

$(\exists x) \neg B(x) \checkmark$

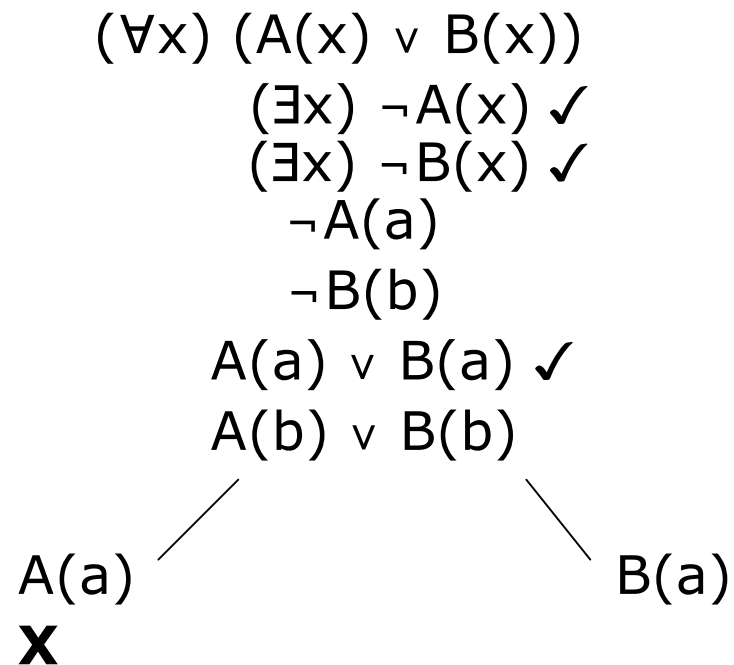
$\neg A(a)$

$\neg B(b)$

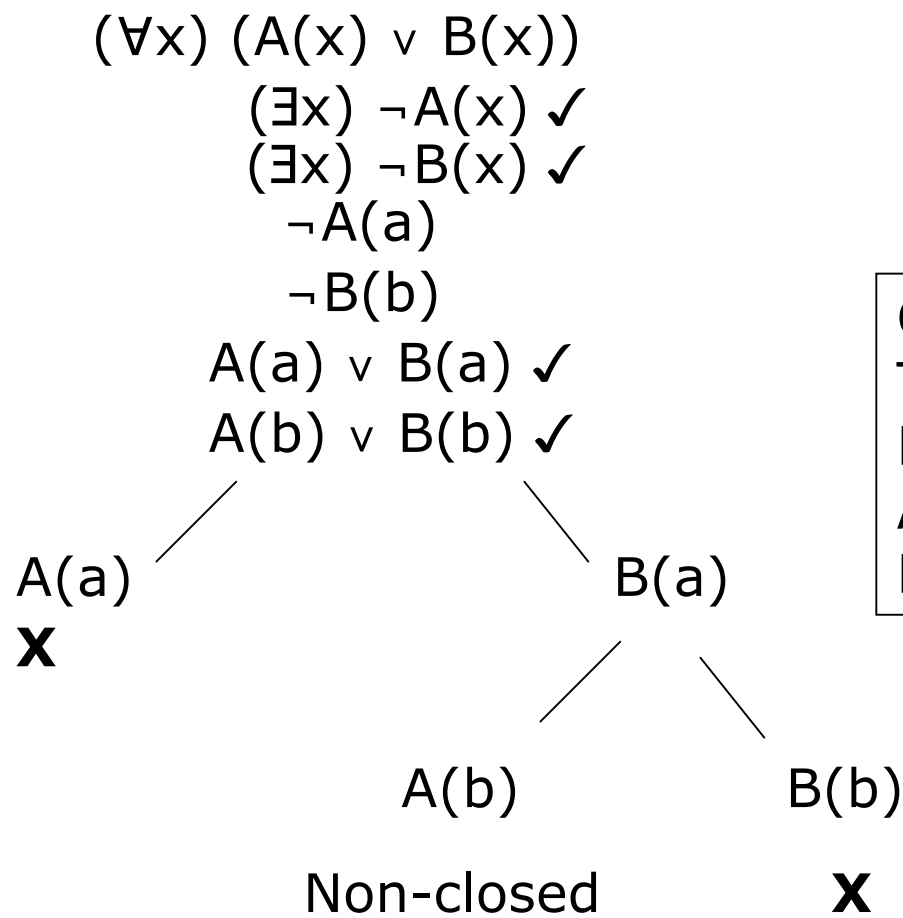
$A(a) \vee B(a)$

$A(b) \vee B(b)$

# Using the Tree Method to Find a Model in the Predicate Calculus



# Using the Tree Method to Find a Model in the Predicate Calculus



Conclusion:  
There is a model  
 $D = \{a, b\}$   
 $A = \{(b)\}$   
 $B = \{(a)\}$