Tableau Method for Predicate Logic

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Aside: Additional Tree Rules: $\varphi \equiv \psi$

- This formula is retired and the tree **branches** with the formulas and their negations stacked:

$$\varphi \equiv \psi \checkmark$$

φ	$\neg\varphi$
ψ	$\neg\psi$

Additional Tree Rules: $\neg(\varphi \equiv \psi)$

- This formula is retired and the tree **branches**, with φ negated:

$$\neg(\varphi \equiv \psi) \checkmark$$

φ	$\neg\varphi$
$\neg\psi$	ψ

Propositional Tree Rule Summary

<table>
<tr><td>stack</td><td>$\varphi \wedge \psi$

φ
ψ</td><td>$\neg(\varphi \vee \psi)$

$\neg\varphi$
$\neg\psi$</td><td>$\neg(\varphi \rightarrow \psi)$

φ
$\neg\psi$</td><td></td><td>$\neg\neg\varphi$

φ</td></tr>
<tr><td rowspan="2">split</td><td rowspan="2">$\varphi \vee \psi$

φ ψ</td><td rowspan="2">$\neg(\varphi \wedge \psi)$

$\neg\varphi$ $\neg\psi$</td><td rowspan="2">$\varphi \rightarrow \psi$

$\neg\varphi$ ψ</td><td>$(\varphi \equiv \psi)$
φ $\neg\varphi$
ψ $\neg\psi$</td><td rowspan="2"></td></tr>
<tr><td>$\neg(\varphi \equiv \psi)$
φ $\neg\varphi$
$\neg\psi$ ψ</td></tr>
</table>

¬∃ rule

$$\neg(\exists \nu)\ \varphi\ ✓$$
$$(\forall \nu)\ \neg\varphi$$

¬∀ rule

$$\neg(\forall \nu)\ \varphi\ ✓$$
$$(\exists \nu)\ \neg\varphi$$

∃ rule

$$(\exists \nu)\ \varphi\ ✓$$

$$\varphi[c/\nu]$$

where c is a **new** constant not appearing in the tree.

The rule can be used only once per ∃ formula.

∀ rule

$(\forall \nu)\ \varphi$ Does not get a check!!

$\varphi[\tau/\nu]$

where τ is any term free to replace ν in φ.

This rule can used arbitrarily-many times for a ∀ formula.

Example

$$\neg((\forall x)p(x) \rightarrow (\exists x)p(x))$$

Example

$$\neg((\forall x)p(x) \rightarrow (\exists x)p(x)) \checkmark$$

$$(\forall x)p(x)$$

$$\neg(\exists x)p(x)$$

Example

$$\neg((\forall x)p(x) \rightarrow (\exists x)p(x)) \checkmark$$

$$(\forall x)p(x)$$

$$\neg(\exists x)p(x) \checkmark$$

$$(\forall x) \neg p(x)$$

Example

$$\neg((\forall x)p(x) \rightarrow (\exists x)p(x)) \checkmark$$

$$(\forall x)p(x) \quad \forall$$

$$\neg(\exists x)p(x) \checkmark$$

$$(\forall x)\neg p(x) \quad \forall$$

$$p(a)$$

$$\neg p(a)$$

X closed

The root formula is not satisfiable.

$(\forall x)p(x) \rightarrow (\exists x)p(x)$ is valid

Closure depended on appropriate choice of term to substitute for x in $(\forall x)\neg p(x)$.

Example

(in which ∀ rule is used twice on the same line)

$$\neg(\exists x)\,(p(x) \rightarrow (\forall y)p(y))$$

Example

$$\neg(\exists x)\ (p(x) \rightarrow (\forall y)p(y))\ ✓$$

$$(\forall x)\ \neg(p(x) \rightarrow (\forall y)p(y))$$

Example

$$\neg(\exists x)\ (p(x) \rightarrow (\forall y)p(y))\ \checkmark$$

$$(\forall x)\ \neg(p(x) \rightarrow (\forall y)p(y))$$

$$\neg(p(a) \rightarrow (\forall y)p(y))$$

Example

$$\neg(\exists x)\ (p(x) \rightarrow (\forall y)p(y))\ \checkmark$$

$$(\forall x)\ \neg(p(x) \rightarrow (\forall y)p(y))$$

$$\neg(p(a) \rightarrow (\forall y)p(y))\ \checkmark$$

$$p(a)$$

$$\neg(\forall y)p(y)$$

Example

$$\neg(\exists x)\,(p(x) \rightarrow (\forall y)p(y))\ \checkmark$$

$$(\forall x)\,\neg(p(x) \rightarrow (\forall y)p(y))$$

$$\neg(p(a) \rightarrow (\forall y)p(y))\ \checkmark$$

$$p(a)$$

$$\neg(\forall y)p(y)\qquad \checkmark$$

$$(\exists y)\,\neg p(y)$$

Example

$\neg(\exists x)\ (p(x) \rightarrow (\forall y)p(y))$ ✓

$(\forall x)\ \neg(p(x) \rightarrow (\forall y)p(y))$

$\neg(p(a) \rightarrow (\forall y)p(y))$ ✓

$p(a)$

$\neg(\forall y)p(y)$ ✓

$(\exists y)\ \neg p(y)$ ✓

$\neg p(b)$

Example

$\neg(\exists x)\,(p(x) \rightarrow (\forall y)p(y))$ ✓

$(\forall x)\,\neg(p(x) \rightarrow (\forall y)p(y))$

$\neg(p(a) \rightarrow (\forall y)p(y))$ ✓

$p(a)$

$\neg(\forall y)p(y)$ ✓

$(\exists y)\,\neg p(y)$ ✓

$\neg p(b)$

$\neg(p(b) \rightarrow (\forall y)p(y))$

Example

$$\neg(\exists x)\,(p(x) \rightarrow (\forall y)p(y))\ \checkmark$$

$$(\forall x)\,\neg(p(x) \rightarrow (\forall y)p(y))$$

$$\neg(p(a) \rightarrow (\forall y)p(y))\ \checkmark$$

$$p(a)$$

$$\neg(\forall y)p(y)\ \checkmark$$

$$(\exists y)\,\neg p(y)\ \checkmark$$

$$\neg p(b)$$

$$\neg(p(b) \rightarrow (\forall y)p(y))\ \checkmark$$

$$p(b)$$

$$\neg(\forall y)p(y)$$

X

closes ($\neg p(b)$ and $p(b)$)

Exercise

- Show that $(\exists x)\ (p(x) \rightarrow (\forall y)p(y))$ is valid (from basic definitions).

- Somehow, this looks wrong:

  In a non-empty crowd,
  there is a person x such that
  (if x is a genius, everyone is a genius.)

  What's the catch?

- Give a natural deduction proof of it.
  (Hint: Use the LEM with $(\forall y)p(y)$.)

Proof of $(\exists x)\ (p(x) \rightarrow (\forall y)p(y))$

1:	actual i	premise
2:	$\forall y.P(y) \vee \neg \forall y.P(y)$	Theorem $E \vee \neg E$
3:	$\forall y.P(y)$	assumption
4:	$P(i) \rightarrow \forall y.P(y)$	Theorem $E \vdash F \rightarrow E$ 3
5:	$\exists x.(P(x) \rightarrow \forall y.P(y))$	$\exists$ intro 4,1
6:	$\neg \forall y.P(y)$	assumption
7:	$\exists z.\neg P(z)$	Theorem $\neg \forall x.R(x) \vdash \exists y.\neg R(y)$ 6
8:	actual i1, $\neg P(i1)$	assumptions
9:	$P(i1) \rightarrow \forall y.P(y)$	Theorem $\neg E \vdash E \rightarrow F$ 8.2
10:	$\exists x.(P(x) \rightarrow \forall y.P(y))$	$\exists$ intro 9,8.1
11:	$\exists x.(P(x) \rightarrow \forall y.P(y))$	$\exists$ elim 7,8-10
12:	$\exists x.(P(x) \rightarrow \forall y.P(y))$	$\vee$ elim 2,3-5,6-11

Needed for non-empty domain assumption

Termination

- Unlike the propositional case, the predicate version of tableaux does not necessarily terminate. This is because the $\forall$ rule can be used arbitrarily-many times.

- It can be shown, however, that **if** the root formula is unsatisfiable, then **there exists** a closed tree for it.
  (If the root formula is satisfiable, the construction *might* not terminate.)

Example of Non-Termination

$$(\forall x)\ (\exists y)\ (A(x) \wedge B(y))$$

Example of Non-Termination

$$(\forall x)\ (\exists y)\ (A(x) \wedge B(y))$$

$$(\exists y)\ (A(a) \wedge B(y))$$

Example of Non-Termination

$$(\forall x)\ (\exists y)\ (A(x) \wedge B(y))$$

$$(\exists y)\ (A(a) \wedge B(y))\ ✓$$

$$(A(a) \wedge B(b_0))$$

Example of Non-Termination

$(\forall x)\ (\exists y)\ (A(x) \wedge B(y))$

$(\exists y)\ (A(a) \wedge B(y))$ ✓

$(A(a) \wedge B(b_0))$ ✓

$A(a)$

$B(b_0)$

Example of Non-Termination

$$(\forall x)\ (\exists y)\ (A(x) \wedge B(y))$$

$$(\exists y)\ (A(a) \wedge B(y))\ \checkmark$$

$$(A(a) \wedge B(b_0))\ \checkmark$$

$$A(a)$$

$$B(b_0)$$

$$(\exists y)\ (A(b_0) \wedge B(y))$$

Example of Non-Termination

$(\forall x)\ (\exists y)\ (A(x) \wedge B(y))$

$(\exists y)\ (A(a) \wedge B(y))$ ✓

$(A(a) \wedge B(b_0))$ ✓

$A(a)$

$B(b_0)$

$(\exists y)\ (A(b_0) \wedge B(y))$ ✓

$A(b_0) \wedge B(b_1)$

Example of Non-Termination

$(\forall x)\ (\exists y)\ (A(x) \wedge B(y))$

$(\exists y)\ (A(a) \wedge B(y))$ ✓

$(A(a) \wedge B(b_0))$ ✓

$A(a)$

$B(b_0)$

$(\exists y)\ (A(b_0) \wedge B(y))$ ✓

$A(b_0) \wedge B(b_1)$ ✓

$A(b_0)$

$B(b_1)$

Example of Non-Termination

$(\forall x)\ (\exists y)\ (A(x) \wedge B(y))$

$(\exists y)\ (A(a) \wedge B(y))$ ✓

$(A(a) \wedge B(b_0))$ ✓

$A(a)$

$B(b_0)$

$(\exists y)\ (A(b_0) \wedge B(y))$ ✓

$A(b_0) \wedge B(b_1)$ ✓

$A(b_0)$

$B(b_1)$

$(\exists y)\ (A(b_1) \wedge B(y))$

…

Example of Non-Termination

$$(\forall x)\,(\exists y)\,(A(x) \wedge B(y))$$

$$(\exists y)\,(A(a) \wedge B(y))\ \checkmark$$

$$(A(a) \wedge B(b_0))\ \checkmark$$

$$A(a)$$

$$B(b_0)$$

We can see at this point that the formula is satisfiable.

A 1-element interpretation will do, with domain $\{0\}$, $\mu(a) = \mu(b_0) = 0$, $A = B = \{(0)\}$.

Using the Tableau Method to Find a Model in the Predicate Calculus

- $(\forall x)\ (A(x) \vee B(x)) \rightarrow ((\forall x)A(x) \vee (\forall x)B(x))$
- This formula is not valid, so its negation should be satisfiable.

$\neg((\forall x)\ (A(x) \vee B(x)) \rightarrow ((\forall x)A(x) \vee (\forall x)B(x)))$ ✓

$(\forall x)\ (A(x) \vee B(x))$

$\neg((\forall x)A(x) \vee (\forall x)B(x))$ ✓

$\neg(\forall x)A(x)$ ✓

$\neg(\forall x)B(x)$ ✓

$(\exists x)\ \neg A(x)$

$(\exists x)\ \neg B(x)$

Using the Tree Method to Find a Model in the Predicate Calculus

- So far we have unchecked:

$$(\forall x)\ (A(x) \lor B(x))$$

$$(\exists x)\ \neg A(x)$$

$$(\exists x)\ \neg B(x)$$

Using the Tree Method to Find a Model in the Predicate Calculus

$(\forall x)\ (A(x) \vee B(x))$

$(\exists x)\ \neg A(x)$ ✓

$(\exists x)\ \neg B(x)$ ✓

$\neg A(a)$

$\neg B(b)$

$A(a) \vee B(a)$

$A(b) \vee B(b)$

Using the Tree Method to Find a Model in the Predicate Calculus

(∀x) (A(x) ∨ B(x))

(∃x) ¬A(x) ✓

(∃x) ¬B(x) ✓

¬A(a)

¬B(b)

A(a) ∨ B(a) ✓

A(b) ∨ B(b)

A(a) B(a)

X

Using the Tree Method to Find a Model in the Predicate Calculus

$(\forall x)\ (A(x) \lor B(x))$

$(\exists x)\ \neg A(x)$ ✓

$(\exists x)\ \neg B(x)$ ✓

$\neg A(a)$

$\neg B(b)$

$A(a) \lor B(a)$ ✓

$A(b) \lor B(b)$ ✓

- $A(a)$
  - **X**
- $B(a)$
  - $A(b)$
    - Non-closed
  - $B(b)$
    - **X**

Conclusion:
There is a model
$D = \{a, b\}$
$A = \{(b)\}$
$B = \{(a)\}$