Meta-Logic: Soundness and Completeness for Propositional Logic

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Meta-?

• “Meta-Logic” or “Meta-Mathematics” means proving things about logic rather than just within logic.

• For example, we might want to prove something about all proofs or all theorems, or that a certain formula is not a theorem.

• The language about which we prove things is called the object language.

• The language within which we prove things about the object language is the meta-language.
Validity vs. Provability

- The symbols $\vdash$ and $\models$ are part of the meta-language.

- $\varphi_1, \ldots, \varphi_n \vdash \psi$ means $\psi$ is provable from $\varphi_1, \ldots, \varphi_n$ (sequent)

- $\varphi_1, \ldots, \varphi_n \models \psi$ means $\varphi_1, \ldots, \varphi_n$ entail $\psi$:
  
  Under any interpretation, if each of $\varphi_i$ is true, then $\psi$ is true.

- $\vdash \psi$ and $\models \psi$ are the special case with $n = 0$. 
Validity vs. Provability

- Generally, if $\Gamma$ is (possibly-infinite) set of formulas
- The symbols $\vdash$ and $\models$ are part of the meta-language.
- $\Gamma \vdash \psi$ means $\psi$ is provable from formulas $\Gamma$
- $\Gamma \models \psi$ means:
  Any interpretation that satisfies (every formula in) $\Gamma$ also satisfies $\psi$.

[An interpretation satisfies a formula if it induces the value $T$.

An interpretation satisfies a set of formulas if it satisfies each formula in the set.]
Satisfiability

- \( \Gamma \) is **satisfiable** if there is an interpretation that satisfies it.

- **Lemma S:** \( \Gamma \) is **satisfiable** iff not \( \Gamma \models \bot \).

- Proof: Suppose that \( \Gamma \) is **satisfiable**. Let \( \nu \) be an interpretation that satisfies it. **No** interpretation satisfies \( \bot \). So it is **not** the case that every interpretation satisfying \( \Gamma \) satisfies \( \bot \), i.e. “not \( \Gamma \models \bot \”).

  Conversely, suppose “not \( \Gamma \models \bot \)”. This says there is **some** valuation which satisfies \( \Gamma \) but does **not** satisfy \( \bot \). But all valuations don’t satisfy \( \bot \). So there is simply some valuation which satisfies \( \Gamma \), i.e. \( \Gamma \) is satisfiable.
Soundness vs. Completeness of a logical system

- **Soundness**: Every provable sequent is an entailment:

  (for every \( \Gamma \) and \( \psi \)):

  \[ \Gamma \vdash \psi \text{ implies } \Gamma \models \psi \]

- **Completeness**: Every valid sequent is provable:

  (for every \( \Gamma \) and \( \psi \)):

  \[ \Gamma \models \psi \text{ implies } \Gamma \vdash \psi \]
Recall Definition of “Truth” for the Propositional Case

• an interpretation is a mapping from proposition symbols \{p, q, r, \ldots\} to the set \{T, F\}.

• an interpretation \(\nu\) is **extended** to an arbitrary formulas inductively as follows:
  • \(\nu(T) = T\).
  • \(\nu(\bot) = F\).
  • \(\nu(\varphi \land \psi) = T\) iff \(\nu(\varphi) = T\) and \(\nu(\psi) = T\).
  • \(\nu(\varphi \lor \psi) = T\) iff \(\nu(\varphi) = T\) or \(\nu(\psi) = T\).
  • \(\nu(\varphi \rightarrow \psi) = T\) iff \(\nu(\varphi) = F\) or \(\nu(\psi) = T\).
  • \(\nu(\neg \varphi) = T\) iff \(\nu(\varphi) = F\).
Proof of Soundness

- **Soundness**: Every sequent of Natural Deduction is an entailment:
  
  (for every $\Gamma$, $\psi$):
  
  $\Gamma \vdash \psi$ implies $\Gamma \models \psi$

  and
  
  $\models \psi$ implies $\models \psi$

- Assume that $\Gamma \vdash \psi$, to show $\Gamma \models \psi$.

- This will be by **structural induction** on the proof tree of $\psi$ from formulas in $\Gamma$. 
Contextual Representation of Natural Deduction Rules

• In the representation of natural deduction rules, the context of premises was assumed.
• For example, with \( \land \) introduction, premises that lead to \( \varphi \) and \( \psi \) in the proof are not shown explicitly.

\[
\begin{array}{c}
\varphi \quad \psi \\
\hline
\varphi \land \psi
\end{array}
\]

\( \land I \)

• For the soundness proof, however, it will be helpful to show the premises explicitly.
• So we restate this rule with contexts as follows:

\[
\begin{array}{c}
\Gamma \quad \Delta \\
\hline
\varphi \\
\Delta, \psi
\end{array}
\]

\( \land I \)

\[
\Gamma, \Delta \quad \varphi \land \psi
\]

Note: \( \Gamma, \Delta \) is shorthand for \( \Gamma \cup \Delta \).
Note that it may be that \( \Gamma = \Delta \), in which case \( \Gamma, \Delta \) is just \( \Gamma \).
Contextual Representation of Natural Deduction Rules

• The contextual form will have its advantages when temporary assumptions are involved, such as in the $\rightarrow$I rule:

$$
\Gamma, \varphi \vdash \psi \\
\Gamma \vdash \varphi \rightarrow \psi \\
\Gamma \vdash \varphi \rightarrow \psi
$$

• Here the notation $\Gamma, \varphi$ is shorthand for $\Gamma \cup \{\varphi\}$. 
<table>
<thead>
<tr>
<th></th>
<th>Introduction</th>
<th>Elimination</th>
</tr>
</thead>
</table>
| \( \wedge \) | \[
\Gamma \vdash \varphi, \\
\Delta \vdash \psi \\
\]
\[
\Gamma, \Delta \vdash \varphi \wedge \psi
\] | \[
\Gamma \vdash \varphi \wedge \psi, \\
\Gamma \vdash \varphi, \\
\Gamma \vdash \psi
\] |\[
\Gamma \vdash \psi
\] |
| \( \lor \) | \[
\Gamma \vdash \varphi, \\
\Gamma \vdash \psi
\] | \[
\Gamma \vdash \varphi \lor \psi, \\
\Gamma, \varphi \vdash \xi, \\
\Gamma, \psi \vdash \xi
\] | \[
\Gamma \vdash \xi
\] |
| \( \to \) | \[
\Gamma, \varphi \vdash \psi
\] | \[
\Gamma \vdash \varphi, \\
\Delta \vdash \varphi \to \psi
\] | \[
\Gamma, \Delta \vdash \psi
\] |
| \( \neg \) | \[
\Gamma, \varphi \vdash \bot
\] | \[
\Gamma \vdash \varphi, \\
\Gamma \vdash \neg \varphi
\] | \[
\Gamma \vdash \bot
\] |
| RAA     | \[
\Gamma, \neg \varphi \vdash \bot
\] | \[
\Gamma, \neg \varphi \vdash \bot
\] |
| \( \bot \) | \[
\Gamma \vdash \bot
\] | \[
\Gamma \vdash \bot
\] |
| T       | \[
\Gamma \vdash T
\] | \[
\Gamma \vdash T
\] |
Example of Box vs. Contextual Form

1: \((E \land F) \rightarrow G\) premise

2: \(E\) assumption

3: \(F\) assumption

4: \(E \land F\) \& intro 2,3

5: \(G\) \rightarrow elim 1,4

6: \(F \rightarrow G\) \rightarrow intro 3–5

7: \(E \rightarrow F \rightarrow G\) \rightarrow intro 2–6

\[
\begin{align*}
E \quad & F \\
\hline
E, F & \rightarrow E \land F \\
\hline
E \land F \rightarrow G, E, F & \rightarrow E \\
E \land F \rightarrow G, E & \rightarrow I \\
E \land F \rightarrow G & \rightarrow I
\end{align*}
\]
Proof of Soundness

• We are proving: $\Gamma \vdash \psi$ implies $\Gamma \models \psi$, meaning if there is a proof of $\psi$ from $\Gamma$, then for any interpretation $\nu$ such that $\nu(\Gamma) = T$, also $\nu(\psi) = T$.

• Structural induction on the tree of the proof, which is either a single node, or a node adding to one or more sub-trees.

• **Basis:** The simplest proof is a tree of **one node**. That node is justified in one of two ways:
  • The consequent of a rule with no antecedent.
  • A premise.

• The only rule of the first kind is $T$ introduction. But $\nu(T) = T$.

• In the second case, a premise must be in $\Gamma$:
  • If for each $\phi$ in $\Gamma$, $\nu(\phi) = T$ then also $\nu(\psi) = T$, since $\psi$ in $\Gamma$.
  • Hence $\Gamma \models \psi$. 
Proof of Soundness Continued

- **Induction Step: Adding a root combining one or more subtrees.**

  Suppose that all of the antecedents of a rule in sequent form satisfy the property. We need to show that the consequent satisfies the property as well.

  - ∧ Introduction rule:
    \[
    \frac{\Gamma \vdash \varphi \quad \Delta \vdash \psi}{\Gamma, \Delta \vdash \varphi \land \psi}
    \]

  - Assume that $\Gamma \vdash \varphi$ implies $\Gamma \models \varphi$, and $\Delta \vdash \psi$ implies $\Delta \models \psi$.

  - We must show that $\Gamma, \Delta \vdash \varphi \land \psi$ implies $\Gamma, \Delta \models \varphi \land \psi$.

  - Assume $\Gamma \vdash \varphi$, $\Delta \vdash \psi$, and thus $\Gamma, \Delta \vdash \varphi \land \psi$.

  - Suppose $\nu(\Gamma \cup \Delta) = T$, to show $\nu(\varphi \land \psi) = T$.
    Then $\nu(\Gamma) = T$ and $\nu(\Delta) = T$, so from the induction hypothesis, $\nu(\varphi) = \nu(\psi) = T$. Thus from the truth table for $\land$, $\nu(\varphi \land \psi) = T$. 
Proof of Soundness Continued

- **Induction Step, Continued:**

  - The steps for $\land E$, $\lor I$, $\rightarrow E$, $\neg E$, $\bot E$ (ones that don’t introduce assumptions) are analogous to that for $\land I$, and are left to the reader.

- $\rightarrow$ Introduction

  $\Gamma, \varphi \vdash \psi$

  $\Gamma \vdash \varphi \rightarrow \psi$

  - Here $\varphi$ is the assumption used in natural deduction, which is discharged at the end of the sub-proof.

- The induction hypothesis is that $\nu(\Gamma \cup \{\varphi\}) = T$ implies $\nu(\psi) = T$.

- We must show $\nu(\Gamma) = T$ implies $\nu(\varphi \rightarrow \psi) = T$.

- Suppose that $\nu(\Gamma) = T$.
  - If $\nu(\varphi) = T$, then $\nu(\Gamma \cup \{\varphi\}) = T$, and from the induction hypothesis, $\nu(\psi) = T$, so $\nu(\varphi \rightarrow \psi) = T$ from the truth table for $\rightarrow$.
  - If $\nu(\varphi) = F$, then $\nu(\varphi \rightarrow \psi) = T$ from the truth table for $\rightarrow$.

- The step for $\lor E$ (which also introduce assumptions) is analogous to the above.
Proof of Soundness Continued

- **Induction Step:**

- RAA

\[
\frac{\Gamma, \neg \varphi \vdash \bot}{\Gamma \vdash \varphi}
\]

- The induction hypothesis is that \( \nu(\Gamma \cup \{\neg \varphi\}) = T \) implies \( \nu(\bot) = T \).
- **But** \( \nu(\bot) = F \) always, so \( \nu(\Gamma \cup \{\neg \varphi\}) = F \).
- We must show \( \nu(\Gamma) = T \) implies \( \nu(\varphi) = T \).
- Suppose that \( \nu(\Gamma) = T \).
- Then \( \nu(\neg \varphi) = F \), from \( \nu(\Gamma \cup \{\neg \varphi\}) = F \).
- Then \( \nu(\varphi) = T \), from the truth table for \( \neg \).
- The step for \( \neg \text{-I} \) is analogous to the above.
- This concludes the proof of the induction step.
Uses of Soundness

- There is an algorithm for determining whether or not
  \[ \varphi_1, \ldots, \varphi_n \models \psi \]

- Thus, one can compute a necessary condition of whether there is a proof of
  \[ \varphi_1, \ldots, \varphi_n \vdash \psi \]

- In other words, before embarking on trying to find a proof of a formula, we can sometimes check whether the formula follows on semantic grounds first.
Completeness

- Completeness says

\[(\text{for all } \Gamma, \psi)\]

\[\Gamma \models \psi \implies \Gamma \vdash \psi\]

- The general case will require a "non-constructive" proof, since \(\Gamma\) could be infinite.

- The case of \(\Gamma\) \textbf{finite} is special, and admits a constructive proof.
Finite Completeness

- Finite completeness says (for all $\varphi_1, \ldots, \varphi_n, \psi$)

$$\varphi_1, \ldots, \varphi_n \vdash \psi$$

implies

$$\varphi_1, \ldots, \varphi_n \models \psi$$

- **If** this could be established, then the algorithm mentioned for soundness would be a necessary and **sufficient** condition for the existence of a proof. Thus provability could be testable algorithmically.

- Our proof will be classical, as it will use LEM.
Proof of Finite Completeness

Three steps are used:

1. \( \varphi_1, \ldots, \varphi_n \models \psi \) implies \( \models (\varphi_1 \rightarrow (\varphi_2 \rightarrow \ldots (\varphi_n \rightarrow \psi )) \ldots) \)

2. For any formula \( \eta \), \( \models \eta \) implies \( \models \eta \).
   [\( \eta \) could be \( (\varphi_1 \rightarrow (\varphi_2 \rightarrow \ldots (\varphi_n \rightarrow \psi )) \ldots) \), for example.]

3. \( \models (\varphi_1 \rightarrow (\varphi_2 \rightarrow \ldots (\varphi_n \rightarrow \psi )) \ldots) \) implies \( \varphi_1, \ldots, \varphi_n \models \psi \)

   **Step 2 is the key one**, as only it bridges the gap between \( \models \) and \( \models \). The other two are simplifying steps, showing that we don’t need to worry about the LHS of the turnstiles.

   Steps 1 and 3 can be proved by induction on \( n \).
Proof that for all $\eta$

$\models \eta$ implies $\vdash \eta$

- The proof is by structural induction on the form of $\eta$.

- Assume $\models \eta$. Let $p_1, p_2, \ldots, p_k$ be the set of all proposition symbols that occur in $\eta$.

- For each combination of proposition symbols with and without negation, we show that there is a sequent with that combination on the left and the formula of interest on the right:
  - $p_1, p_2, \ldots, p_k \vdash \eta$
  - $\neg p_1, p_2, \ldots, p_k \vdash \eta$
  - $p_1, \neg p_2, \ldots, p_k \vdash \eta$
  - $\neg p_1, \neg p_2, \ldots, p_k \vdash \eta$
    etc.

- Then those sequents will be combined into a single sequent of the required form.
The Combination Process

- Because this constructs a derivation that is of length exponential in \( k \), we will show it by example, for \( k = 2 \).

- Given that we have:
  - \( p_1, p_2 \vdash \eta \)
  - \( \neg p_1, p_2 \vdash \eta \)
  - \( p_1, \neg p_2 \vdash \eta \)
  - \( \neg p_1, \neg p_2 \vdash \eta \)

- The proof constructed for the single sequent is shown on the next page.
Proof Constructed for the Single Sequent

<table>
<thead>
<tr>
<th>Step</th>
<th>Formula</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$p_1 \lor \neg p_1$</td>
<td>LEM (a derived rule)</td>
</tr>
<tr>
<td>2.</td>
<td>$p_1$</td>
<td>Assumption</td>
</tr>
<tr>
<td>3.</td>
<td>$p_2 \lor \neg p_2$</td>
<td>LEM</td>
</tr>
<tr>
<td>4.</td>
<td>$p_2$</td>
<td>Assumption</td>
</tr>
<tr>
<td></td>
<td>\ldots steps in the proof of $p_1 \land p_2 \vdash \eta$</td>
<td></td>
</tr>
<tr>
<td>5.</td>
<td>$\eta$</td>
<td></td>
</tr>
<tr>
<td>6.</td>
<td>$\neg p_2$</td>
<td>Assumption</td>
</tr>
<tr>
<td></td>
<td>\ldots steps in the proof of $p_1 \land \neg p_2 \vdash \eta$</td>
<td></td>
</tr>
<tr>
<td>7.</td>
<td>$\eta$</td>
<td></td>
</tr>
<tr>
<td>8.</td>
<td>$\eta$</td>
<td>$\lor E$ 3, 4-5, 6-7</td>
</tr>
<tr>
<td>9.</td>
<td>$\neg p_1$</td>
<td>Assumption</td>
</tr>
<tr>
<td>10.</td>
<td>$p_2 \lor \neg p_2$</td>
<td>LEM</td>
</tr>
<tr>
<td>11.</td>
<td>$p_2$</td>
<td>Assumption</td>
</tr>
<tr>
<td></td>
<td>\ldots steps in the proof of $\neg p_1 \land p_2 \vdash \eta$</td>
<td></td>
</tr>
<tr>
<td>12.</td>
<td>$\eta$</td>
<td></td>
</tr>
<tr>
<td>13.</td>
<td>$\neg p_2$</td>
<td>Assumption</td>
</tr>
<tr>
<td></td>
<td>\ldots steps in the proof of $\neg p_1 \land \neg p_2 \vdash \eta$</td>
<td></td>
</tr>
<tr>
<td>14.</td>
<td>$\eta$</td>
<td></td>
</tr>
<tr>
<td>15.</td>
<td>$\eta$</td>
<td>$\lor E$ 10, 11-12, 13-14</td>
</tr>
<tr>
<td>16.</td>
<td>$\eta$</td>
<td>$\lor E$ 1, 2-8, 9-15</td>
</tr>
</tbody>
</table>
Proofs for the Individual Sequents

- We are left with showing that each of the individual sequents
  - \( p_1, p_2, \ldots, p_k \vdash \eta \)
  - \( \neg p_1, p_2, \ldots, p_k \vdash \eta \)
  - \( p_1, \neg p_2, \ldots, p_k \vdash \eta \)
  - \( \neg p_1, \neg p_2, \ldots, p_k \vdash \eta \)
  - etc.

has a proof, given that
- \( \models \eta \).
Proofs for the Individual Sequents

• For any formula $\eta$, we want to show that $|\eta|$ implies each of the individual sequents below has a proof
  - $p_1, p_2, \ldots, p_k \vdash \eta$
  - $\neg p_1, p_2, \ldots, p_k \vdash \eta$
  - $p_1, \neg p_2, \ldots, p_k \vdash \eta$
  - $\neg p_1, \neg p_2, \ldots, p_k \vdash \eta$ etc.
where $p_1, p_2, \ldots, p_k$ are the proposition symbols in $\eta$.

• Consider any combination $p^*_1, p^*_2, \ldots, p^*_k$ of the symbols negated or un-negated (e.g. $\neg p_1, p_2, \ldots, \neg p_k$) and the corresponding **assignment** that makes $\nu(p^*_1 \land p^*_2 \land \ldots \land p^*_k) = T$.

• **Lemma:**
  A: If $\nu(\eta) = T$, then $p^*_1, p^*_2, \ldots, p^*_k \vdash \eta$.
  
  B: If $\nu(\eta) = F$, then $p^*_1, p^*_2, \ldots, p^*_k \vdash (\neg \eta)$. Note the $\neg$. 
Proving

A. If \( \nu(\eta) = T \) then \( p^*_1, p^*_2, \ldots, p^*_k \models \eta \).

B. If \( \nu(\eta) = F \) then \( p^*_1, p^*_2, \ldots, p^*_k \models (\neg \eta) \).

- This is done by **structural induction** on the **structure** of the **formula** \( \eta \).

- **Basis**: If \( \eta \) is a **single proposition symbol** \( p \), then:
  - If \( \nu(p) = T \), then \( p^* \) must be \( p \), and we certainly have \( p \models p \) (so A).
  - If \( \nu(p) = F \), then \( p^* \) must be \( \neg p \), and we have \( \neg p \models \neg p \) (so B).
  - If \( \eta \) is \( \bot \), then \( \nu(\bot) = F \) always, but also \( \models \neg \bot \) (by \( \neg I \))(so B).

- **Induction Step**: We have to show that the inductive hypothesis implies the conclusion for each possible operator: \( \neg \land \lor \rightarrow \).
Case where $\eta$ is of form $\neg \rho$ for some $\rho$:

- If $\nu(\eta) = T$, then $\nu(\rho) = F$. By the induction hypothesis, B:
  \[ p^*_1, p^*_2, \ldots, p^*_k \vdash (\neg \rho), \]
  i.e.
  \[ p^*_1, p^*_2, \ldots, p^*_k \vdash \eta, \text{ (case A)}. \]

- If $\nu(\eta) = F$, then $\nu(\rho) = T$. By the induction hypothesis, A:
  \[ p^*_1, p^*_2, \ldots, p^*_k \vdash \rho. \]

Using $\neg
\neg$-I to extend the proof one step, we have
\[ p^*_1, p^*_2, \ldots, p^*_k \vdash \neg (\neg \rho). \]
Therefore
\[ p^*_1, p^*_2, \ldots, p^*_k \vdash \neg \eta, \text{ (case B)}. \]
Case where $\eta$ is of form $\rho_1 \land \rho_2$

- We need to consider 4 cases: 
  $\nu(\rho_1, \rho_2) = FF, FT, TF, \text{ and } TT$.

- If $\nu(\rho_1) = F$: 
  By the induction hypothesis 
  
  $p_{1}^{*}, p_{2}^{*}, \ldots, p_{k}^{*} \vdash (\neg \rho_1)$ 

  Using ND rules, we get, in a few more ND steps, a proof of 
  
  $p_{1}^{*}, p_{2}^{*}, \ldots, p_{k}^{*} \vdash \neg (\rho_1 \land \rho_2)$ 

  This conforms to the fact that $\nu(\rho_1 \land \rho_2) = F$ (case B).

- A similar argument applies if $\nu(\rho_2) = F$.

- If $\nu(\rho_1) = \nu(\rho_2) = T$, we have by the induction hypothesis 
  
  $p_{1}^{*}, p_{2}^{*}, \ldots, p_{k}^{*} \vdash \rho_1$ 

  $p_{1}^{*}, p_{2}^{*}, \ldots, p_{k}^{*} \vdash \rho_2$ 

  These proofs can be combined using $\land I$ to get a proof of $\rho_1 \land \rho_2$ (case A).

- The cases for the other operators ($\lor, \rightarrow$) are similar.
Algorithm-Based Proof

- The proof just outlined is sufficiently constructive that we can create an algorithm from it:
  - Given a tautology $\eta$, generate a natural deduction proof of $\eta$.
  - In some sense, such an algorithmic proof is useful, in that it can be actively tested for examples, unlike an ordinary proof.
Completeness in the General Propositional Case: Consistency

- **Definition**: A set of formulas $\Gamma$ is consistent provided
  \[ \neg \Gamma \vdash \bot. \]

- Note the parallel:
  - **Consistency** of $\Gamma$: Not $\Gamma \vdash \bot$.
  - **Satisfiability** of $\Gamma$: Not $\Gamma \models \bot$. 
Lemma A

• For any $\Gamma, \varphi$

\[ \Gamma \models \varphi \quad \text{iff} \quad \Gamma \cup \{\neg \varphi\} \models \bot \]

• Proof:
  
  • Suppose $\Gamma \models \varphi$. Then $\Gamma \cup \{\neg \varphi\} \models \varphi$
  (since if $\Gamma \subseteq \Gamma'$ and $\Gamma \models \varphi$, then $\Gamma' \models \varphi$).
  Also $\Gamma \cup \{\neg \varphi\} \models \neg \varphi$. So $\Gamma \cup \{\neg \varphi\} \models \bot$ by $\neg \text{E}$.

  • Suppose $\Gamma \cup \{\neg \varphi\} \models \bot$. Then by RAA, $\Gamma \models \varphi$. 
Lemma B

• For any $\Gamma$, $\varphi$

$$\Gamma \models \varphi \iff \Gamma \cup \{\neg \varphi\} \models \bot.$$  

• Proof:
  
  • Suppose $\Gamma \models \varphi$. Suppose $\nu$ is an assignment such that $\nu(\Gamma \cup \{\neg \varphi\}) = T$. Then $\nu(\Gamma) = T$ and $\nu(\neg \varphi) = T$. But, by the supposition, also $\nu(\varphi) = T$, giving a contradiction. Thus there is thus no such $\nu$, i.e. $\Gamma \cup \{\neg \varphi\} \models \bot$.

  • Conversely, suppose $\Gamma \cup \{\neg \varphi\} \models \bot$. Then there is no assignment satisfying both $\Gamma$ and $\neg \varphi$. Thus any assignment $\nu$ satisfying $\Gamma$ cannot also satisfy $\neg \varphi$, i.e. $\nu(\neg \varphi) = F$, giving $\nu(\varphi) = T$. Hence $\Gamma \models \varphi$. 
Lemma C

- The following are equivalent:
  a) Completeness.
  b) For all $\Gamma$, for all $\varphi$, $\Gamma \models \varphi$ implies $\Gamma \vdash \varphi$.
  c) For all $\Gamma$, for all $\varphi$, $\Gamma \models \bot$ implies $\Gamma \vdash \bot$.
  d) For all $\Gamma$, for all $\varphi$, not $\Gamma \vdash \bot$ implies not $\Gamma \models \bot$.
  e) For all $\Gamma$, $\Gamma$ is consistent implies $\Gamma$ has a model.

- Proof:
  - (b) is a restatement of (a).
  - (c) iff (b) is by Lemmas A and B.
  - (d) is the contrapositive of (c).
  - (e) is a restatement of (d).
General Completeness Theorem
(for $\Gamma$ not-necessarily finite)

- We have shown that completeness is equivalent to:

- (For all $\Gamma$)
  $\Gamma$ consistent implies $\Gamma$ satisfiable.

- Sketch of the proof of the above:

  We start with a $\Gamma_0$ that is consistent, to eventually show there is an interpretation satisfying $\Gamma_0$.
  The rules will get used in showing this.
Sketch, continued

- First we extend $\Gamma_0$ to a maximally consistent set $\Gamma_{\text{max}}$:
  - Let $\Gamma$ be $\Gamma_0$.
  - Enumerate every possible formula $\varphi$. At each step:
    - If $\Gamma \cup \{\varphi\}$ is consistent, $\Gamma$ becomes $\Gamma \cup \{\varphi\}$.
    - The limit of this process is $\Gamma_{\text{max}}$, the union of the chain of individual consistent sets.
- Then show that $\Gamma_{\text{max}}$ is consistent, and in fact, maximally consistent.
Sketch, continued

• $\Gamma_{\text{max}}$ is consistent, because at no step did we add a formula that would destroy its consistency.

• It is maximally consistent because it can be shown to be closed under derivability:
  
  If $\Gamma_{\text{max}} \vdash \varphi$, then in fact $\varphi \in \Gamma_{\text{max}}$.

• We then show that any maximally consistent set has an interpretation satisfying it. Define an interpretation $\nu$ as follows:
  
  - For each proposition symbol $p$, if $p \in \Gamma_{\text{max}}$ then $\nu(p) = T$, otherwise $\nu(p) = F$.

• Then argue that $\nu$ satisfies $\Gamma_{\text{max}}$ using closure under derivability.

• Finally $\nu$ also satisfies $\Gamma_0$, since $\Gamma_0 \subseteq \Gamma_{\text{max}}$.
Predicate Calculus form of the Completeness Theorem

- Parallels the propositional case in many ways:
  - Construct a maximally-consistent set from the original consistent set.
  - Construct a model for the maximally-consistent set.
How to construct a model out of nothing but formulas?

- Define a universe (called the **Herbrand Universe**), the members of which are terms.

- Example: If there is one constant symbol $c$ and one binary function symbol symbol $f$, then the Herbrand universe is

$$\{c, f(c, c), f(c, f(c, c)), f(f(c, c), c), \ldots \}$$
Model construction, continued

- Create an interpretation with the Herbrand universe as its universe:
  - The function symbols are interpreted in the “obvious” way.
  - The predicate symbols are interpreted so as to agree with atomic formulas in the maximally-consistent set.
  - New constants (called Henkin-constants) are introduced into the language to provide constants that solve ∃-formulas. However, these constants do not change the derivability of expressions in the original language.
  - It is then shown that this interpretation is a model for the original formulas.
First-Order Theories vs. Frameworks

• The natural deduction framework is both sound and complete.

• However, there are first-order theories that are incomplete.

• This may be the source of some confusion.
Theories

• A **theory** is a set of formulas closed under derivability. The formulas are called **theorems**.

• Usually the set is based on a smaller set of **axioms**.

• The set of axioms may be:
  • Finite
  • Computable
  • or neither, but the first two are the most useful.
Examples of a Theory

- Peano axioms
- Peano arithmetic (PA), included +, x
- Theory of groups
- Set theory (e.g. ZFC)
Completeness of a Theory

- A *theory* is **complete** if, for any closed formula, either:
  - The formula is provable, or
  - The negation of the formula is provable.

- Sometimes, this form of completeness is called “negation-completeness”, to separate it from the completeness of the framework.
Gödel’s Theorems

• Gödel was the first to show that first-order logic is a **complete** framework: Every valid (true in all interpretations) statement is provable.

• Gödel later showed that number theory is **incomplete** (provided the theory itself is consistent):

There are true statements in number theory that are not provable in that theory.

Gödel’s proof showed how to construct such statements.

• Gödel then showed that the **consistency** of number theory can’t be proved within number theory itself ("second incompleteness theorem").
Decidable Theories

- A theory is **decidable** if there is an algorithm that will tell whether or not any given formula is a theorem.

- Complete (+ recursively axiomatized) $\Rightarrow$ Decidable

- A corollary of Gödel’s first incompleteness theorem is that number theory is **undecidable**. (because it is recursively-axiomatized, but incomplete).