Meta-Logic: Soundness and Completeness for Propositional Logic

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Meta-?

- “Meta-Logic” or “Meta-Mathematics”
  means proving things ***about*** logic rather than just ***within*** logic.

- For example, we might want to prove something about all proofs or all theorems, or that a certain formula is ***not*** a theorem.

- The language ***about*** which we prove things is called the **object** language.

- The language ***within*** which we prove things about the object language is the **meta-language**.

Validity vs. Provability

- The symbols $\vdash$ and $\models$ are part of the meta-language.
- $\varphi_1, \ldots, \varphi_n \vdash \psi$ means ψ is **provable** from $\varphi_1, \ldots, \varphi_n$ (sequent)
- $\varphi_1, \ldots, \varphi_n \models \psi$ means $\varphi_1, \ldots, \varphi_n$ **entail** ψ:

  Under any interpretation, if each of φ_i is true, then ψ is true.
- $\vdash \psi$ and $\models \psi$ are the special case with $n = 0$.

Validity vs. Provability

- Generally, if Γ is (possibly-infinite) set of formulas
- The symbols $\vdash$ and $\models$ are part of the meta-language.
- $\Gamma \vdash \psi$ means ψ is **provable** from formulas Γ
- $\Gamma \models \psi$ means:

  Any interpretation that satisfies (every formula in) Γ also satisfies ψ.

  [An interpretation **satisfies a formula** if it induces the value T.

  An interpretation **satisfies a set** of formulas if it satisfies each formula in the set.]

Satisfiability

- Γ is **satisfiable** if there is an interpretation that satisfies it.
- **Lemma S**: Γ is **satisfiable** iff not $\Gamma \models \bot$.
- Proof: Suppose that Γ is **satisfiable**. Let ν be an interpretation that satisfies it. ***No*** interpretation satisfies $\bot$. So it is ***not*** the case that every interpretation satisfying Γ satisfies $\bot$, i.e. "not $\Gamma \models \bot$".

  Conversely, suppose "not $\Gamma \models \bot$". This says there is ***some*** valuation which satisfies Γ but does ***not*** satisfy $\bot$. But all valuations don't satisfy $\bot$. So there is simply some valuation which satisfies Γ, i.e. Γ is satisfiable.

Soundness vs. Completeness of a logical system

- **Soundness**: Every provable sequent is an entailment:

  (for every Γ and ψ):

  $\Gamma \vdash \psi$ implies $\Gamma \models \psi$

- **Completeness**: Every valid sequent is provable:

  (for every Γ and ψ):

  $\Gamma \models \psi$ implies $\Gamma \vdash \psi$

Recall Definition of “Truth” for the Propositional Case

- an interpretation is a mapping from proposition symbols {p, q, r, …} to the set {**T**, **F**}.

- an interpretation ν is **extended** to an arbitrary formulas inductively as follows:
    - $\nu(T) = \mathbf{T}$.
    - $\nu(\bot) = \mathbf{F}$.
    - $\nu(\varphi \wedge \psi) = \mathbf{T}$ iff $\nu(\varphi) = \mathbf{T}$ and $\nu(\psi) = \mathbf{T}$.
    - $\nu(\varphi \vee \psi) = \mathbf{T}$ iff $\nu(\varphi) = \mathbf{T}$ or $\nu(\psi) = \mathbf{T}$.
    - $\nu(\varphi \rightarrow \psi) = \mathbf{T}$ iff $\nu(\varphi) = \mathbf{F}$ or $\nu(\psi) = \mathbf{T}$.
    - $\nu(\neg \varphi) = \mathbf{T}$ iff $\nu(\varphi) = \mathbf{F}$.

Proof of Soundness

- **Soundness**: Every sequent of Natural Deduction is an entailment:

  (for every Γ, ψ):

  $$\Gamma \vdash \psi \text{ implies } \Gamma \models \psi$$

  and $\vdash \psi$ implies $\models \psi$

- Assume that $\Gamma \vdash \psi$, to show $\Gamma \models \psi$.
- This will be by **structural induction** on the proof tree of ψ from formulas in Γ.

Contextual Representation of Natural Deduction Rules

- In the representation of natural deduction rules, the **context** of premises was assumed.
- For example, with ∧ introduction, premises that lead to φ and ψ in the proof are not shown explicitly.

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \quad \wedge I$$

- For the soundness proof, however, it will be helpful to show the premises explicitly.
- So we restate this rule **with contexts** as follows:

$$\frac{\Gamma \vdash \varphi \qquad \Delta \vdash \psi}{\Gamma, \Delta \vdash \varphi \wedge \psi} \quad \wedge I$$

Note: Γ, Δ is shorthand for $\Gamma \cup \Delta$.

Note that it may be that $\Gamma = \Delta$, in which case Γ, Δ is just Γ.

Contextual Representation of Natural Deduction Rules

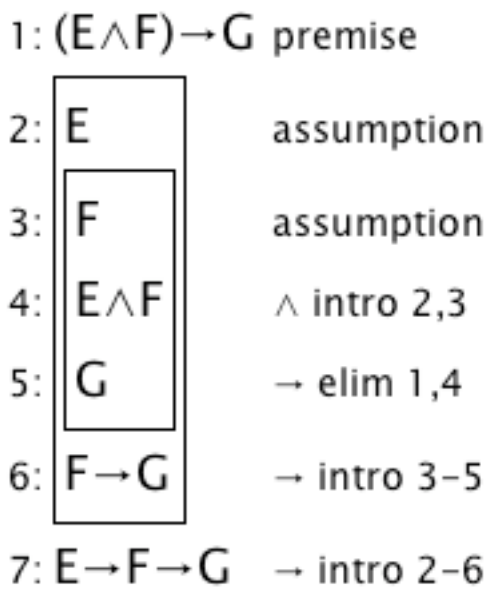
- The contextual form will have its advantages when temporary assumptions are involved, such as in the →I rule:

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \rightarrow \psi} \quad \rightarrow I$$

- Here the notation Γ, φ is shorthand for $\Gamma \cup \{\varphi\}$.

ND Rules in Contextual Representation		
	Introduction	**Elimination**
∧	$\dfrac{\Gamma \vdash \varphi \qquad \Delta \vdash \psi}{\Gamma,\Delta \vdash \varphi\wedge\psi}$	$\dfrac{\Gamma \vdash \varphi\wedge\psi}{\Gamma \vdash \varphi}$ $\qquad$ $\dfrac{\Gamma \vdash \varphi\wedge\psi}{\Gamma \vdash \psi}$
∨	$\dfrac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi\vee\psi}$ $\qquad$ $\dfrac{\Gamma \vdash \psi}{\Gamma \vdash \varphi\vee\psi}$	$\dfrac{\Gamma \vdash \varphi\vee\psi \qquad \Gamma, \varphi \vdash \xi \qquad \Gamma, \psi \vdash \xi}{\Gamma \vdash \xi}$
→	$\dfrac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi\to\psi}$	$\dfrac{\Gamma \vdash \varphi \qquad \Delta \vdash \varphi\to\psi}{\Gamma,\Delta \vdash \psi}$
¬	$\dfrac{\Gamma, \varphi \vdash \bot}{\Gamma \vdash \neg\varphi}$	$\dfrac{\Gamma \vdash \varphi \qquad \Gamma \vdash \neg\varphi}{\Gamma \vdash \bot}$
RAA ⊥E	$\dfrac{\Gamma, \neg\varphi \vdash \bot}{\Gamma \vdash \varphi}$	$\dfrac{}{\Gamma, \bot \vdash \varphi}$
T	$\dfrac{}{\Gamma \vdash \mathrm{T}}$	

Example of Box vs. Contextual Form



1: (E∧F)→G premise

2: E assumption

3: F assumption

4: E∧F ∧ intro 2,3

5: G → elim 1,4

6: F→G → intro 3-5

7: E→F→G → intro 2-6

$$\dfrac{\dfrac{\dfrac{\dfrac{E \vdash E \qquad F \vdash F}{E, F \vdash E\land F}\ \land I \qquad\qquad E\land F\to G \vdash E\land F\to G}{(E\land F\to G), E, F \vdash G}\ \to E}{(E\land F\to G), E \vdash F\to G}\ \to I}{(E\land F\to G) \vdash E\to F\to G}\ \to I$$

Contextual Rule Applications

Proof of Soundness

- We are proving: $\Gamma \vdash \psi$ implies $\Gamma \models \psi$, meaning if there is a proof of ψ from Γ, then for any interpretation ν such that $\nu(\Gamma)$ = T, also $\nu(\psi)$ = T.

- **Structural induction on the tree of the proof, which is either a single node, or a node adding to one or more sub-trees.**

- **Basis**: The simplest proof is a tree of **one node**. That node is justified in one of two ways:
  - The consequent of a rule with no antecedent.
  - A premise.

- The only rule of the first kind is T introduction. But ν(T) = T.
- In the second case, a premise must be in Γ:
  - If for each φ in Γ, $\nu(\varphi)$ = **T** then also $\nu(\psi)$ = **T**, since ψ in Γ.
  - Hence $\Gamma \models \psi$.

Proof of Soundness Continued

- **Induction Step: Adding a root combining one or more subtrees.**
- Suppose that all of the antecedents of a rule in sequent form satisfy the property. We need to show that the consequent satisfies the property as well.
- $\wedge$ Introduction rule:

$$\frac{\Gamma \vdash \varphi \qquad \Delta \vdash \psi}{\Gamma,\, \Delta \vdash \varphi \wedge \psi}$$

- Assume that $\Gamma \vdash \varphi$ implies $\Gamma \models \varphi$, and $\Delta \vdash \psi$ implies $\Delta \models \psi$.
- We must show that $\Gamma, \Delta \vdash \varphi \wedge \psi$ implies $\Gamma, \Delta \models \varphi \wedge \psi$.
- Assume $\Gamma \vdash \varphi$, $\Delta \vdash \psi$, and thus $\Gamma, \Delta \vdash \varphi \wedge \psi$.
- Suppose $\nu(\Gamma \cup \Delta) = T$, to show $\nu(\varphi \wedge \psi) = T$.
  Then $\nu(\Gamma) = T$ and $\nu(\Delta) = T$, so
  from the induction hypothesis, $\nu(\varphi) = \nu(\psi) = T$.
  Thus from the truth table for $\wedge$, $\nu(\varphi \wedge \psi) = T$.

Proof of Soundness Continued

- **Induction Step, Continued:**
- The steps for ∧E, ∨I, →E, ¬E, ⊥E (ones that don't introduce assumptions) are analogous to that for ∧I, and are left to the reader.
- → Introduction

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \rightarrow \psi}$$

- Here φ is the assumption used in natural deduction, which is discharged at the end of the sub-proof.
- The induction hypothesis is that $\nu(\Gamma \cup \{\varphi\}) = T$ implies $\nu(\psi) = T$.
- We must show $\nu(\Gamma) = T$ implies $\nu(\varphi \rightarrow \psi) = T$.
- Suppose that $\nu(\Gamma) = T$.
  - If $\nu(\varphi) = T$, then $\nu(\Gamma \cup \{\varphi\}) = T$, and from the induction hypothesis, $\nu(\psi) = T$, so $\nu(\varphi \rightarrow \psi) = T$ from the truth table for →.
  - If $\nu(\varphi) = F$, then $\nu(\varphi \rightarrow \psi) = T$ from the truth table for →.
- The step for ∨E (which also introduce assumptions) is analogous to the above.

Proof of Soundness Continued

- **Induction Step:**
- RAA

$$\frac{\Gamma, \neg\varphi \vdash \bot}{\Gamma \vdash \varphi}$$

- The induction hypothesis is that $\nu(\Gamma \cup \{\neg\varphi\}) = T$ implies $\nu(\bot) = T$.
- But $\nu(\bot) = F$ always, so $\nu(\Gamma \cup \{\neg\varphi\}) = F$.
- We must show $\nu(\Gamma) = T$ implies $\nu(\varphi) = T$.
- Suppose that $\nu(\Gamma) = T$.
- Then $\nu(\neg\varphi) = F$, from $\nu(\Gamma \cup \{\neg\varphi\}) = F$.
- Then $\nu(\varphi) = T$, from the truth table for $\neg$.
- The step for $\neg I$ is analogous to the above.
- This concludes the proof of the induction step.

Uses of Soundness

- There is an algorithm for determining whether or not

$$\varphi_1, \ldots, \varphi_n \models \psi$$

- Thus, one can compute a **necessary** condition of whether there is a proof of

$$\varphi_1, \ldots, \varphi_n \vdash \psi$$

- In other words, before embarking on trying to find a proof of a formula, we can sometimes check whether the formula follows on semantic grounds first.

Completeness

- Completeness says

  (for all Γ, ψ)

  $$\Gamma \models \psi \text{ implies } \Gamma \vdash \psi$$

- The general case will require a “non-constructive” proof, since Γ could be infinite.
- The case of Γ **finite** is special, and admits a constructive proof.

Finite Completeness

- Finite completeness says (for all $\varphi_1, \ldots, \varphi_n, \psi$)

  $$\varphi_1, \ldots, \varphi_n \models \psi$$

  implies

  $$\varphi_1, \ldots, \varphi_n \vdash \psi$$

- **If** this could be established, then the algorithm mentioned for soundness would be a necessary and **sufficient** condition for the existence of a proof. Thus provability could be testable algorithmically.

- Our proof will be classical, as it will use LEM.

Proof of Finite Completeness

Three steps are used:

1. $\varphi_1, \ldots, \varphi_n \models \psi$ implies $\models (\varphi_1 \rightarrow (\varphi_2 \rightarrow \ldots (\varphi_n \rightarrow \psi)) \ldots)$
2. For any formula η, $\models \eta$ implies $\vdash \eta$.
   [η could be $(\varphi_1 \rightarrow (\varphi_2 \rightarrow \ldots (\varphi_n \rightarrow \psi)) \ldots)$, for example.]
3. $\vdash (\varphi_1 \rightarrow (\varphi_2 \rightarrow \ldots (\varphi_n \rightarrow \psi)) \ldots)$ implies $\varphi_1, \ldots, \varphi_n \vdash \psi$

Step 2 is the key one, as only it bridges the gap between $\models$ and $\vdash$. The other two are simplifying steps, showing that we don't need to worry about the LHS of the turnstiles.

Steps 1 and 3 can be proved by induction on n.

Proof that for all η

$$\models \eta \text{ implies } \vdash \eta$$

- The proof is by structural induction on the form of η.
- Assume $\models \eta$. Let $p_1, p_2, \ldots, p_k$ be the set of all proposition symbols that occur in η.
- For each combination of proposition symbols with and without negation, we show that there is a sequent with that combination on the left and the formula of interest on the right:
  - $p_1, \; p_2, \ldots, p_k \vdash \eta$
  - $\neg p_1, \; p_2, \ldots, p_k \vdash \eta$
  - $p_1, \neg p_2, \ldots, p_k \vdash \eta$
  - $\neg p_1, \neg p_2, \ldots, p_k \vdash \eta$ etc.
- Then those sequents will be combined into a single sequent of the required form.

The Combination Process

- Because this constructs a derivation that is of length exponential in k, we will show it by example, for k = 2.

- Given that we have:
  - $p_1, \; p_2 \vdash \eta$
  - $\neg p_1, \; p_2 \vdash \eta$
  - $p_1, \neg p_2 \vdash \eta$
  - $\neg p_1, \neg p_2 \vdash \eta$

- The proof constructed for the single sequent is shown on the next page.

Proof Constructed for the Single Sequent

1.	$p_1 \vee \neg p_1$	LEM (a derived rule)
2.	p_1	Assumption
3.	$p_2 \vee \neg p_2$	LEM
4.	p_2	Assumption
	. . . steps in the proof of $\mathbf{p_1, p_2} \vdash \eta$	
5.	η	
6.	$\neg p_2$	Assumption
	. . . steps in the proof of $\mathbf{p_1, \neg p_2} \vdash \eta$	
7.	η	
8.	η	$\vee$E 3, 4-5, 6-7
9.	$\neg p_1$	Assumption
10.	$p_2 \vee \neg p_2$	LEM
11.	p_2	Assumption
	. . . steps in the proof of $\mathbf{\neg p_1, p_2} \vdash \eta$	
12.	η	
13.	$\neg p_2$	Assumption
	. . . steps in the proof of $\mathbf{\neg p_1, \neg p_2} \vdash \eta$	
14.	η	
15.	η	$\vee$E 10, 11-12, 13-14
16.	η	$\vee$E 1, 2-8, 9-15

Proofs for the Individual Sequents

- We are left with showing that each of the individual sequents
  - $p_1, \quad p_2, \ldots, p_k \vdash \eta$
  - $\neg p_1, \quad p_2, \ldots, p_k \vdash \eta$
  - $p_1, \neg p_2, \ldots, p_k \vdash \eta$
  - $\neg p_1, \neg p_2, \ldots, p_k \vdash \eta$ etc.

  has a proof, given that
  - $\models \eta$.

Proofs for the Individual Sequents

- For any formula η, we want to show that $\models \eta$ implies each of the individual sequents below has a proof
  - $p_1, \quad p_2, \ldots, p_k \vdash \eta$
  - $\neg p_1, \quad p_2, \ldots, p_k \vdash \eta$
  - $p_1, \neg p_2, \ldots, p_k \vdash \eta$
  - $\neg p_1, \neg p_2, \ldots, p_k \vdash \eta$ etc.

  where $p_1, p_2, \ldots, p_k$ are the proposition symbols in η.

- Consider any combination $p^*_1, p^*_2, \ldots, p^*_k$ of the symbols negated or un-negated (e.g. $\neg p_1, p_2, \ldots, \neg p_k$) and the corresponding **assignment** that makes $\nu(p^*_1 \wedge p^*_2 \wedge \ldots \wedge p^*_k) = \mathbf{T}$.

- **Lemma:**

  A: If $\nu(\eta) = \mathbf{T}$, then $p^*_1, p^*_2, \ldots, p^*_k \vdash \eta$.

  B: If $\nu(\eta) = \mathbf{F}$, then $p^*_1, p^*_2, \ldots, p^*_k \vdash (\neg\eta)$. Note the $\neg$.

Proving A. If $\nu(\eta)$ = **T** then $p^*_1, p^*_2, \ldots, p^*_k \vdash \eta$.

B. If $\nu(\eta)$ = **F** then $p^*_1, p^*_2, \ldots, p^*_k \vdash (\neg\eta)$.

- This is done by **structural induction** on the **structure** of the **formula** η.
- **Basis**: If η is a **single proposition symbol** p, then:
  - If $\nu(p)$ = **T**, then p* must be p, and we certainly have $p \vdash p$ (so A).
  - If $\nu(p)$ = **F** , then p* must be $\neg p$, and we have $\neg p \vdash \neg p$ (so B).
  - If η is $\bot$, then $\nu(\bot)$ = **F** always, but also $\vdash \neg\bot$ (by $\neg$I)(so B).
- **Induction Step**: We have to show that the inductive hypothesis implies the conclusion for each possible operator: $\neg \wedge \vee \rightarrow$.

Case where η is of form $\neg\rho$ for some ρ :

- If $\nu(\eta)$ = **T**, then $\nu(\rho)$ = **F**. By the induction hypothesis, B:

  $$p^*_1, p^*_2, \ldots, p^*_k \vdash (\neg\rho),$$

  i.e.

  $$p^*_1, p^*_2, \ldots, p^*_k \vdash \eta, \text{ (case A).}$$

- If $\nu(\eta)$ = **F**, then $\nu(\rho)$ = **T**. By the induction hypothesis, A:

  $$p^*_1, p^*_2, \ldots, p^*_k \vdash \rho.$$

  Using **¬¬I** to extend the proof one step, we have

  $$p^*_1, p^*_2, \ldots, p^*_k \vdash \neg(\neg\rho).$$

  Therefore

  $$p^*_1, p^*_2, \ldots, p^*_k \vdash \neg\eta, \text{ (case B).}$$

Case where η is of form $\rho_1 \wedge \rho_2$

- We need to consider 4 cases:
  $\nu(\rho_1 , \rho_2)$ = **FF**, **FT**, **TF**, and **TT**.
- If $\nu(\rho_1)$ = **F**:
  By the induction hypothesis
  $p^*_1, p^*_2, . . ., p^*_k \vdash (\neg\rho_1)$
  Using ND rules, we get, in a few more ND steps, a proof of
  $p^*_1, p^*_2, . . ., p^*_k \vdash \neg(\rho_1 \wedge \rho_2)$
  This conforms to the fact that $\nu(\rho_1 \wedge \rho_2)$ = **F** (case B).
- A similar argument applies if $\nu(\rho_2)$ = **F**.

- If $\nu(\rho_1) = \nu(\rho_2)$ = **T,** we have by the induction hypothesis
  $p^*_1, p^*_2, . . ., p^*_k \vdash \rho_1$
  $p^*_1, p^*_2, . . ., p^*_k \vdash \rho_2$
  These proofs can be combined using $\wedge$I to get a proof of $\rho_1 \wedge \rho_2$ (case A).
- **The cases for the other operators ($\vee$, $\rightarrow$) are similar.**

Algorithm-Based Proof

- The proof just outlined is sufficiently constructive that we can create an algorithm from it:
- Given a tautology η, generate a natural deduction proof of η.
- In some sense, such an algorithmic proof is useful, in that it can be actively tested for examples, unlike an ordinary proof.

Completeness in the General Propositional Case: Consistency

- **Definition**: A set of formulas Γ is **consistent** provided

$$\text{not} \quad \Gamma \vdash \bot.$$

- Note the parallel:
  - **Consistency** of Γ: Not $\Gamma \vdash \bot$.
  - **Satisfiability** of Γ: Not $\Gamma \models \bot$.

Lemma A

- For any Γ, φ

  $\Gamma \mid\!\!- \varphi$ iff $\Gamma \cup \{\neg\varphi\} \mid\!\!- \bot$

- Proof:
  - Suppose $\Gamma \mid\!\!- \varphi$. Then $\Gamma \cup \{\neg\varphi\} \mid\!\!- \varphi$
    (since if $\Gamma \subseteq \Gamma'$ and $\Gamma \mid\!\!- \varphi$, then $\Gamma' \mid\!\!- \varphi$).
    Also $\Gamma \cup \{\neg\varphi\} \mid\!\!- \neg\varphi$. So $\Gamma \cup \{\neg\varphi\} \mid\!\!- \bot$ by $\neg$E.
  - Suppose $\Gamma \cup \{\neg\varphi\} \mid\!\!- \bot$. Then by RAA, $\Gamma \mid\!\!- \varphi$.

Lemma B

- For any Γ, φ

  $\Gamma \models \varphi$ iff $\Gamma \cup \{\neg\varphi\} \models \bot$.

- Proof:
  - Suppose $\Gamma \models \varphi$. Suppose ν is an assignment such that $\nu(\Gamma \cup \{\neg\varphi\}) = T$. Then $\nu(\Gamma) = T$ and $\nu(\neg\varphi) = T$. But, by the supposition, also $\nu(\varphi) = T$, giving a contradiction.

    Thus there is thus no such ν, i.e. $\Gamma \cup \{\neg\varphi\} \models \bot$.

  - Conversely, suppose $\Gamma \cup \{\neg\varphi\} \models \bot$. Then there is no assignment satisfying both Γ and $\neg\varphi$. Thus any assignment ν satisfying Γ cannot also satisfy $\neg\varphi$, i.e. $\nu(\neg\varphi) = F$, giving $\nu(\varphi) = T$. Hence $\Gamma \models \varphi$.

Lemma C

- The following are equivalent:
  a) Completeness.
  b) For all Γ, for all φ, $\Gamma \mid= \varphi$ implies $\Gamma \mid- \varphi$.
  c) For all Γ, for all φ, $\Gamma \mid= \perp$ implies $\Gamma \mid- \perp$.
  d) For all Γ, for all φ, not $\Gamma \mid- \perp$ implies not $\Gamma \mid= \perp$.
  e) For all Γ, Γ is consistent implies Γ has a model.

- Proof:
  - (b) is a restatement of (a).
  - (c) iff (b) is by Lemmas A and B.
  - (d) is the contrapositive of (c).
  - (e) is a restatement of (d).

General Completeness Theorem

(for Γ not-necessarily finite)

- We have shown that completeness is equivalent to:

- (For all Γ)
  Γ consistent implies Γ satisfiable.

- Sketch of the proof of the above:

  We start with a Γ_0 that is consistent, to eventually show there is an interpretation satisfying Γ_0.

  The rules will get used in showing this.

Sketch, continued

- First we extend Γ_0 to a maximally consistent set Γ_{max}:
  - Let Γ be Γ_0.

  - Enumerate every possible formula φ. At each step:
    - If $\Gamma \cup \{\varphi\}$ is consistent, Γ becomes $\Gamma \cup \{\varphi\}$.
  - The limit of this process is Γ_{max}, the union of the chain of individual consistent sets.

- Then show that Γ_{max} is consistent, and in fact, maximally consistent.

Sketch, continued

- Γ_{max} **is consistent**, because at no step did we add a formula that would destroy its consistency.

- It is maximally consistent because it can be shown to be **closed under derivability**:

  If $\Gamma_{\text{max}} \vdash \varphi$, then in fact $\varphi \in \Gamma_{\text{max}}$.

- We then show that any maximally consistent set has an interpretation satisfying it. **Define an interpretation** ν as follows:
  - For each proposition symbol p, if $p \in \Gamma_{\text{max}}$ then $\nu(p) = T$, otherwise $\nu(p) = F$.

- Then argue that ν satisfies Γ_{max} using closure under derivability.

- Finally ν also satisfies Γ_0, since $\Gamma_0 \subseteq \Gamma_{\text{max}}$.

Predicate Calculus form of the Completeness Theorem

- Parallels the propositional case in many ways:
  - Construct a maximally-consistent set from the original consistent set.
  - Construct a model for the maximally-consistent set.

How to construct a model out of nothing but formulas?

- Define a universe (called the **Herbrand Universe**), the members of which are terms.

- Example: If there is one constant symbol c and one binary function symbol f, then the Herbrand universe is

  {c, f(c, c), f(c, f(c, c)), f(f(c, c), c), … }

Model construction, continued

- Create an interpretation with the Herbrand universe as its universe:
  - The function symbols are interpreted in the “obvious” way.
  - The predicate symbols are interpreted so as to **agree with atomic formulas** in the maximally-consistent set.
  - New constants (called **Henkin-constants**) are introduced into the language to provide constants that **solve ∃-formulas**. However, these constants do not change the derivability of expressions in the original language.
  - It is then shown that this interpretation is a model for the original formulas.

First-Order Theories vs. Frameworks

- The natural deduction **framework** is both sound and complete.
- However, there are first-order **theories** that are incomplete.
- This may be the source of some confusion.

Theories

- A **theory** is a set of formulas closed under derivability. The formulas are called **theorems**.

- Usually the set is based on a smaller set of **axioms**.

- The set of axioms may be:
    - Finite
    - Computable
    - or neither, but the first two are the most useful.

Examples of a Theory

- Peano axioms
- Peano arithmetic (PA), included +, x
- Theory of groups
- Set theory (e.g. ZFC)

Completeness of a Theory

- A *theory* is **complete** if, for any closed formula, either:
    - The formula is provable, or
    - The negation of the formula is provable.
- Sometimes, this form of completeness is called “negation-completeness”, to separate it from the completeness of the framework.

Gödel’s Theorems

- Gödel was the first to show that first-order logic is a **complete** framework: Every valid (true in all interpretations) statement is provable.

- Gödel later showed that number theory is **incomplete** (provided the theory itself is consistent):

  There are true statements in number theory that are not provable in that theory.

  Gödel’s proof showed how to construct such statements.

- Gödel then showed that the **consistency** of number theory can’t be proved within number theory itself (“second incompleteness theorem”).

Decidable Theories

- A theory is **decidable** if there is an algorithm that will tell whether or not any given formula is a theorem.
- Complete (+ recursively axiomatized) ⇒ Decidable
- A corollary of Gödel’s first incompleteness theorem is that number theory is **undecidable**. (because it is recursively-axiomatized, but incomplete).