Sequent Calculus

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Sequent Calculus (SC) (Gentzen System S)

- The sequent calculus was used by Gentzen to derive results about Natural Deduction. It generalizes his system G by allowing **sets** of formulas on **both** sides of |–.

- In the SC, |– becomes an **object-**language symbol, rather than a meta-language symbol.

- The *intuitive* meaning of |– is:

  $A_1, \ldots, A_m \vdash B_1, \ldots, B_n$ is like

  $(A_1 \wedge \ldots \wedge A_m) \rightarrow (B_1 \vee \ldots \vee B_n)$ [yes, ∨ on the right] which is equivalent to

  $\neg A_1 \vee \ldots \vee \neg A_m \vee B_1 \vee \ldots \vee B_n$

  So one needs to pay careful attention to left vs. right.

  The case of n = 1 is typically of most interest for the last step.

Ben-Ari’s Presentation of S

- Ben-Ari uses $\Rightarrow$ instead of |–
to emphasize that, in S, $\Rightarrow$ is part of the object language, rather than the meta-language. We don’t, in these slides.

- Some other authors and JAPE use |– anyway.

- Smullyan and Prawitz use $\rightarrow$ instead of |–, and use $\supset$ for implication.

Sequent Calculus Rules

(Ben-Ari version)

Definition 3.46 Axioms in the Gentzen sequent system $\mathcal{S}$ are sequents of the form: $U \cup \{A\} \Rightarrow V \cup \{A\}$. The rules of inference are:

op	Introduction into consequent	Introduction into antecedent
$\wedge$	$\dfrac{U \Rightarrow V \cup \{A\} \qquad U \Rightarrow V \cup \{B\}}{U \Rightarrow V \cup \{A \wedge B\}}$	$\dfrac{U \cup \{A, B\} \Rightarrow V}{U \cup \{A \wedge B\} \Rightarrow V}$
$\vee$	$\dfrac{U \Rightarrow V \cup \{A, B\}}{U \Rightarrow V \cup \{A \vee B\}}$	$\dfrac{U \cup \{A\} \Rightarrow V \qquad U \cup \{B\} \Rightarrow V}{U \cup \{A \vee B\} \Rightarrow V}$
$\rightarrow$	$\dfrac{U \cup \{A\} \Rightarrow V \cup \{B\}}{U \Rightarrow V \cup \{A \rightarrow B\}}$	$\dfrac{U \Rightarrow V \cup \{A\} \qquad U \cup \{B\} \Rightarrow V}{U \cup \{A \rightarrow B\} \Rightarrow V}$
$\neg$	$\dfrac{U \cup \{A\} \Rightarrow V}{U \Rightarrow V \cup \{\neg A\}}$	$\dfrac{U \Rightarrow V \cup \{A\}}{U \cup \{\neg A\} \Rightarrow V}$

Sequent Calculus vs. ND

Definition 3.46 Axioms in the Gentzen sequent system S are sequents of the form: $U \cup \{A\} \Rightarrow V \cup \{A\}$. The rules of inference are:

op	Introduction into consequent	Introduction into antecedent
$\wedge$	essentially ∧I $\dfrac{U \Rightarrow V \cup \{A\} \qquad U \Rightarrow V \cup \{B\}}{U \Rightarrow V \cup \{A \wedge B\}}$	essentially ∧E $\dfrac{U \cup \{A, B\} \Rightarrow V}{U \cup \{A \wedge B\} \Rightarrow V}$
$\vee$	essentially ∨I $\dfrac{U \Rightarrow V \cup \{A, B\}}{U \Rightarrow V \cup \{A \vee B\}}$	essentially ∨E $\dfrac{U \cup \{A\} \Rightarrow V \qquad U \cup \{B\} \Rightarrow V}{U \cup \{A \vee B\} \Rightarrow V}$
$\rightarrow$	essentially →I $\dfrac{U \cup \{A\} \Rightarrow V \cup \{B\}}{U \Rightarrow V \cup \{A \rightarrow B\}}$	essentially →E $\dfrac{U \Rightarrow V \cup \{A\} \qquad U \cup \{B\} \Rightarrow V}{U \cup \{A \rightarrow B\} \Rightarrow V}$
$\neg$	related to ¬I $\dfrac{U \cup \{A\} \Rightarrow V}{U \Rightarrow V \cup \{\neg A\}}$	related to ¬E $\dfrac{U \Rightarrow V \cup \{A\}}{U \cup \{\neg A\} \Rightarrow V}$

Example Sequent Calculus Proof

Constructed from bottom to top:

$$\dfrac{\dfrac{\dfrac{}{P, P \vdash Q}\ \text{Axiom} \qquad \dfrac{}{Q, \vdash P, Q}\ \text{Axiom}}{P \vee Q \vdash P, Q}\ \vee\vdash \text{rule}}{P \vee Q, \neg P \vdash Q}\ \neg\vdash \text{rule}$$

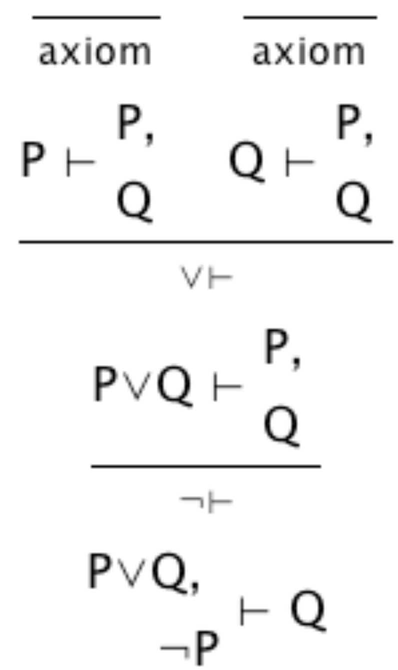
JAPE Version (MCS)

_________ Axiom　　　　　_________ Axiom

P, P |– Q　　　　　　　　Q, |– P, Q　　v |– rule

P v Q |– P, Q　　　　　¬ |– rule

P v Q, ¬P |–Q



MCS = Multi-Conclusion
SCS = Single Conclusion
(SCS is for intuitionistic)

Rough Correspondence: ND vs. SC

JAPE ND:

1: (E∧F)→G premise

2: E assumption

3: F assumption

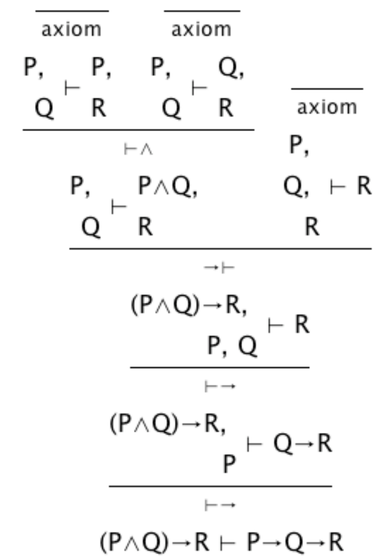
4: E∧F ∧ intro 2,3

5: G → elim 1,4

6: F→G → intro 3-5

7: E→F→G → intro 2-6

JAPE:



From presentation on soundness:

E |— E F |— F

E, F |— E∧F ∧I E∧F→G |— E∧F→G

(E∧F→G), E, F |— G →E

(E∧F→G), E |— F→G →I

(E∧F→G) |— E→F→G →I

Contextual Rule Applications

Rough Correspondence: SC vs. ND

- The last sequent of an SC derivation will always correspond to the overall sequent derived in ND.
- Other sequents may correspond to subproofs.
- Consider **LHS** to be all operative hypotheses, including assumptions in sub-proofs.
- Consider RHS to be goal (as a disjunction).

Hint For the Sequent Calculus

- We are almost always going to be working **backwards**.
- Sequent rules behave like an “abacus”, shifting beads from one side to the other.
- In some case, we need to “split” the beads, and their sequents.
- The leaves of the “sequent tree” (arising from splitting) must all be axioms for the proof to be complete.

More Sequent Calculus Examples
(constructed working backward/upward)

- ___________ Ax ____________ Ax
- _________________ Ax P |– Q, R, P Q, P |– Q, R → |-
- (P → Q), R, P |– R (P → Q), P |– Q, R → |-
- (P → Q), (Q → R), P |– R |- →
- (P → Q), (Q → R) |– (P → R) ∧ |-
- (P → Q) ∧ (Q → R) |– (P → R) |- →
- |– ((P → Q) ∧ (Q → R)) → (P → R)

Previous Example using a spreadsheet
(constructed working backward/downward)

		|–	
1.		((P → Q) ∧ (Q → R)) → (P → R)	
2.	(P → Q) ∧ (Q → R)	P → R	|- →, 1
3.	(P → Q), (Q → R)	P → R	∧ |-, 2
4.	(P → Q), (Q → R), P	R	|- →, 3
5. Ax	(P → Q), R, P	R	→ |-, 4
6.	(P → Q), P	Q, R	→ |-, 4
7. Ax	Q, P	Q, R	→ |-, 6
8. Ax	P	P, Q, R	→ |-, 6

In a complete proof, each line is either an axiom or is used to produce other lines.

More Sequent Calculus Examples
(constructed working upward/backward)

- __________ Ax
- P, Q |– Q, R |- →
- P |– Q, Q → R |- →
- |– P → Q, Q → R |- ∨
- |– (P → Q) ∨ (Q → R)

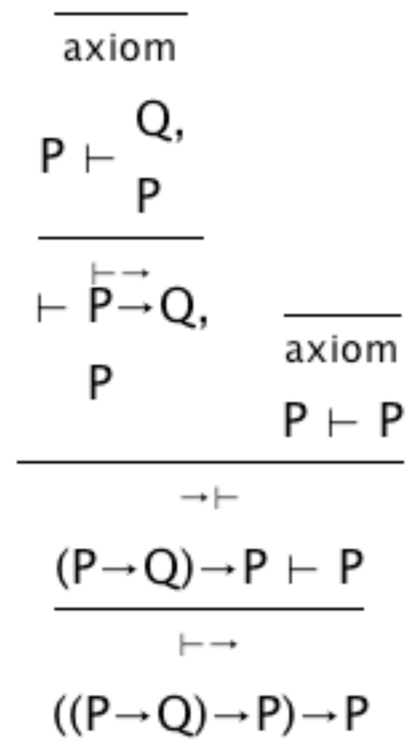
JAPE ND Proof for comparison
(slightly different ordering)

1:	F∨¬F	Theorem E∨¬E
2:	F	assumption
3:	E→F	Theorem E ⊢ F→E 2
4:	(E→F)∨(F→G)	∨ intro 3
5:	¬F	assumption
6:	F→G	Theorem ¬E ⊢ E→F 5
7:	(E→F)∨(F→G)	∨ intro 6
8:	(E→F)∨(F→G)	∨ elim 1,2–4,5–7

More Sequent Calculus Examples
(constructed working upward/backward)

- ________ Ax
- P |– P, Q   ______ Ax
- |– P, P → Q |-→   P |– P
- (P → Q) → P |– P |- →
- |– ((P → Q) → P) → P

JAPE version



Schematic Before/After
Sequent Calculus Rules working backward

Operator **Operator in Left Set** **Operator in Right Set**

∧

Before	___, A∧B, ___ |– …	
After	___, A, B, ___ |– …	

Before	… |– ___, A∧B, ___	
After	… |– __, A, __	… |– __, B, ___

∨

Before	___, A∨B, ___ |– …	
After	__, A, __ |– …	__, B, __ |– …

Before	… |– ___, A∨B, ___
After	… |– ___, A, B, ___

→

Before	___, A→ B, ___ |– …	
After	__, B, __ |– …	__, __ |– …, A

Before	… |– __, A→B, ___
After	…, A |– ___, B, ___

¬

Before	___, ¬A ___ |– …
After	___, ___ |– A …

Before	… |– ___, ¬A ___
After	…, ¬A |– ___, ___

Axiom	___, A, ___ |– …, A, …

Sequent Calculus Informal Summary

Working **backward** from the desired sequent:

Situation	Action
$A \wedge B$ in a right formula	Split right set into A and B versions.
$A \wedge B$ in a left formula	Replace the formula with A, B.
$A \vee B$ in a right formula	Replace the formula with A, B.
$A \vee B$ in a left formula	Split left set into A and B versions.
$\neg A$ in a right formula	Replace with A on the left.
$\neg A$ in a left formula	Replace with A on the right.
$A \rightarrow B$ in a right formula	Replace with A on the left, B on the right.
$A \rightarrow B$ in a left formula	Split into two versions with A on the right in one, B on the left in the other.
A on left and right	Axiom.

An Automated Sequent Calculus Prover:

http://bach.istc.kobe-u.ac.jp/seqprover/

Sequent Prover (seqprover)

Sequent: (p->q)/\(q->r) --> p->r Prove

Output style: Pretty form

[Top page]

```
Trying to prove with threshold = 0

----------- Ax  ----------- Ax   ----------- Ax  ----------- Ax
p --> q,p,r     p,r --> p,r      p,q --> q,r     p,q,r --> r
--------------------------- L->  --------------------------- L->
      p,q->r --> p,r                   p,q,q->r --> r
      ----------------------------------------------- L->
                    p,p->q,q->r --> r
                    ------------------ R->
                    p->q,q->r --> p->r
                    ----------------------- L/\
                    (p->q)/\(q->r) --> p->r

# Proved in 0 msec.
```

SC Rules for Quantifiers

(see http://sakharov.net/sequent.html)

Universal Quantifier	$\dfrac{A(t),\ \Gamma \mid\!- \Delta}{\forall x\ A(x),\ \Gamma \mid\!- \Delta}$	$\dfrac{\Gamma \mid\!- \Delta,\ A(a)}{\Gamma \mid\!- \Delta,\ \forall x\ A(x)}$
Existential Quantifier	$\dfrac{A(a),\ \Gamma \mid\!- \Delta}{\exists x\ A(x),\ \Gamma \mid\!- \Delta}$	$\dfrac{\Gamma \mid\!- \Delta,\ A(t)}{\Gamma \mid\!- \Delta,\ \exists x\ A(x)}$

In these inference rules, Γ, Δ, Σ, Π are finite lists of formulas, A and B are any formulas, x, a, t are any variables. If terms are allowed, then t is any term. A(a) and A(t) are obtained from A(x) by replacing all free occurrences of x by a and t, respectively. The variables a and t are free in A(a) and A(t), respectively. If t is a term, then all variable occurrences in it are free in A(t). The variable a, which occurs in the ∀-succedent rule and the ∃−antecedent rule may not occur free in the lower sequents of the respective rules.

JAPE SC Proof using Quantifiers

$$\dfrac{}{P(m,m1) \vdash P(m,m1)}\ \text{axiom}$$

$$\dfrac{P(m,m1) \vdash P(m,m1)}{P(m,m1) \vdash \exists x.P(x,m1)}\ \vdash\exists$$

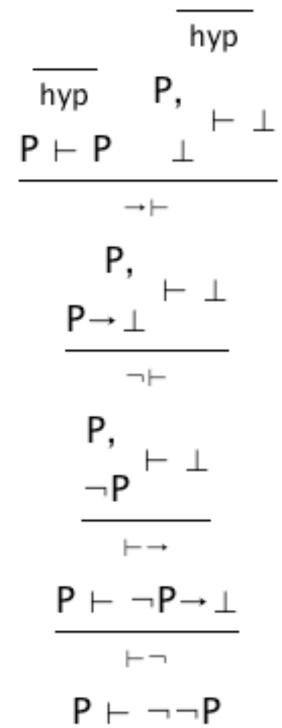
$$\dfrac{P(m,m1) \vdash \exists x.P(x,m1)}{\forall y.P(m,y) \vdash \exists x.P(x,m1)}\ \forall\vdash$$

$$\dfrac{\forall y.P(m,y) \vdash \exists x.P(x,m1)}{\forall y.P(m,y) \vdash \forall y.\exists x.P(x,y)}\ \vdash\forall$$

$$\dfrac{\forall y.P(m,y) \vdash \forall y.\exists x.P(x,y)}{\exists x.\forall y.P(x,y) \vdash \forall y.\exists x.P(x,y)}\ \exists\vdash$$

JAPE SCS vs. MCS

- SCS: Single-conclusion:
  - Will only prove intuitionistic sequents.
  - The RHS is always a single formula.
  - ¬φ converts to φ→⊥ first.



SC vs. Tableaux

- A correspondence can be established between an SC proof and a tableau proof T.
- T is constructed top-down; SC is bottom-up.
- The conclusion in SC is negated in T.

$\neg(\neg A_1 \vee \ldots \vee \neg A_m \vee B_1 \vee \ldots \vee B_n)$ ¬SC

$\equiv A_1 \wedge \ldots \wedge A_m \wedge \neg B_1 \wedge \ldots \wedge \neg B_n$ T

- Formulas on the **right** in SC are like the **negated** formulas in T.
- Formulas on the **left** in SC are like the **un-negated** formulas in T.
- Closure in T is like an axiom in SC.

SC vs. Tableaux

Sequent Calculus	**Tableau**
Constructed bottom-up.	Constructed top-down.
Conclusion is $(\neg A_1 \vee \ldots \vee \neg A_m \vee B_1 \vee \ldots \vee B_n)$	Negated conclusion is $\neg(\neg A_1 \vee \ldots \vee \neg A_m \vee B_1 \vee \ldots \vee B_n)$ $\equiv A_1 \wedge \ldots \wedge A_m \wedge \neg B_1 \wedge \ldots \wedge \neg B_n$
Formulas on the right	Negated formulas
Formulas on the left	Un-negated formulas
Axiom …, p, … |- __, p, __	Closure p, ¬p

Sequent Calculus vs. Tableau

The conclusion in SC is on the **right** (negated in T).

Working **backward** from the desired sequent:

SC	Tableau	Action
$A \wedge B$ in a right formula	$\neg(A \wedge B)$	Split right set into A and B versions.
$A \wedge B$ in a left formula	$A \wedge B$	Replace the formula with A, B.
$A \vee B$ in a right formula	$\neg(A \vee B)$	Replace the formula with A, B.
$A \vee B$ in a left formula	$A \vee B$	Split left set into A and B versions.
$\neg A$ in a right formula	use appropriate rule	Replace with A on the left.
$\neg A$ in a left formula	$\neg\neg A$	Replace with A on the right.
$A \rightarrow B$ in a right formula	$\neg(A \rightarrow B)$	Replace with A on the left, B on the right.
$A \rightarrow B$ in a left formula	$A \rightarrow B$	Split into two versions with A on the **right** in one, B on the **left** in the other.
A on left and right	$\neg A$, A	A |-A Axiom.

Tableau vs. SC Example

Tableau
$\neg((p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p))$

SC "upside-down"
$\vdash (p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$

Tableau vs. SC Example

Tableau
$\neg((p \to q) \to (\neg q \to \neg p))$
$(p \to q)$ $\neg(\neg q \to \neg p)$

SC "upside-down"
$\vdash (p \to q) \to (\neg q \to \neg p)$
$(p \to q) \vdash (\neg q \to \neg p)$

Tableau vs. SC Example

Tableau	SC "upside-down"
$\neg((p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p))$	$\vdash (p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$
$(p \rightarrow q)$ $\neg(\neg q \rightarrow \neg p)$	$(p \rightarrow q) \vdash (\neg q \rightarrow \neg p)$
$\neg q$ $\neg\neg p$	$\neg q, (p \rightarrow q) \vdash \neg p$

Tableau vs. SC Example

Tableau	SC “upside-down”
¬((p → q) → (¬q → ¬p))	|– (p → q) → (¬q → ¬p)
(p → q) ¬(¬q → ¬p)	(p → q) |– (¬q → ¬p)
¬q ¬¬p	¬q, (p → q) |– ¬p
p	p, ¬q, (p → q) |–

Tableau vs. SC Example

Tableau	
$\neg((p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p))$	
$(p \rightarrow q)$ $\neg(\neg q \rightarrow \neg p)$	
$\neg q$ $\neg\neg p$	
p	
$\neg p$	q

SC “upside-down”	
$\vdash (p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$	
$(p \rightarrow q) \vdash (\neg q \rightarrow \neg p)$	
$\neg q, (p \rightarrow q) \vdash \neg p$	
$p, \neg q, (\mathbf{p} \rightarrow \mathbf{q}) \vdash$	
$p, \neg q, \mathbf{q} \vdash$	$p, \neg q \vdash \mathbf{p}$

Tableau vs. SC Example

Tableau	
$\neg((p \to q) \to (\neg q \to \neg p))$	
$(p \to q)$ $\neg(\neg q \to \neg p)$	
$\neg q$ $\neg\neg p$	
p	
$\neg p$	q
X	X

SC "upside-down"	
$\vdash (p \to q) \to (\neg q \to \neg p)$	
$(p \to q) \vdash (\neg q \to \neg p)$	
$\neg q, (p \to q) \vdash \neg p$	
$p, \neg q, (p \to q) \vdash$	
$p, \boldsymbol{\neg q}, q \vdash$	$p, \boldsymbol{\neg q} \vdash p$
$p, q \vdash q$	$p \vdash p, q$

Tableau Prover

http://www.umsu.de/logik/trees/

Please use for checking, not doing, homework!