Adalines

Primitive Artificial Neurons

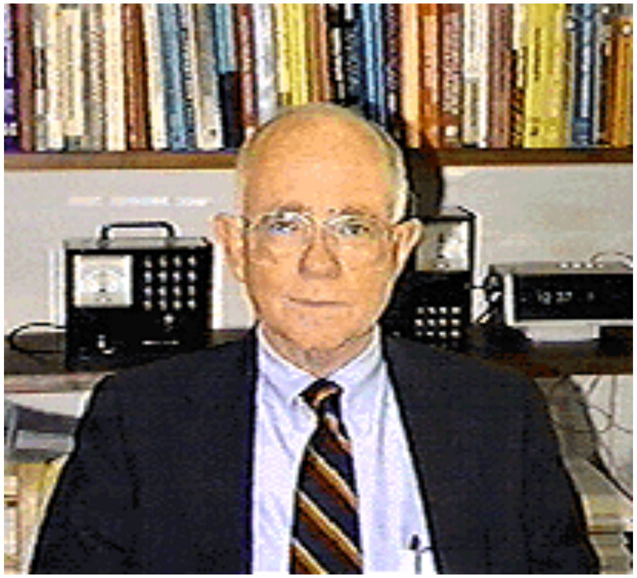
Building blocks for Multi-Layer Networks

(called "multi-layer perceptrons" (MLP) strangely enough)

Sweet Adaline

- The Adaline ("Adaptive Linear Neuron" or "Adaptive Linear Element") is a model similar to the Perceptron.
- There are several variations:
  - One has the **threshold** function same as a perceptron.
  - Another uses a **pure linear** function with no threshold.
  - Others can be devised.

Adaline Inventors
Bernard Widrow and Marcian (Ted) Hoff

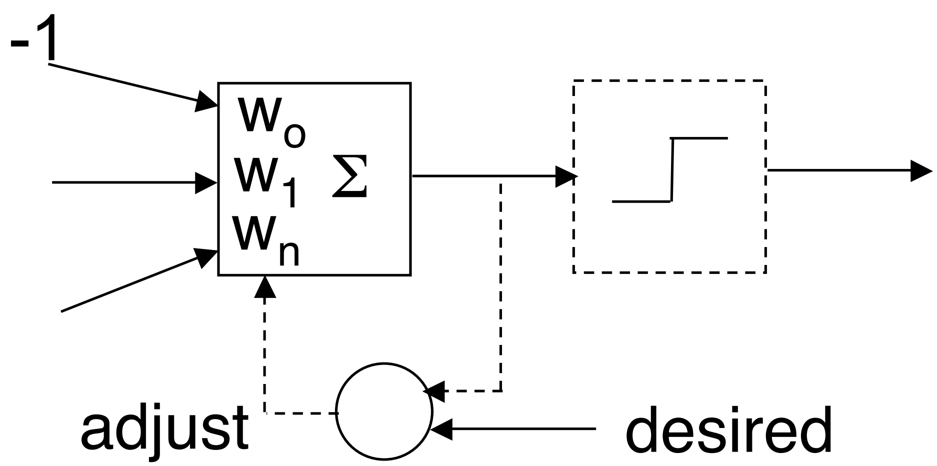


Bernard Widrow,
Professor Emeritus of E.E.,
Stanford University

Marcian Hoff
Co-inventor of Patent 3,821,715
Microprocessor Concept and Architecture

Adaline Training (1)

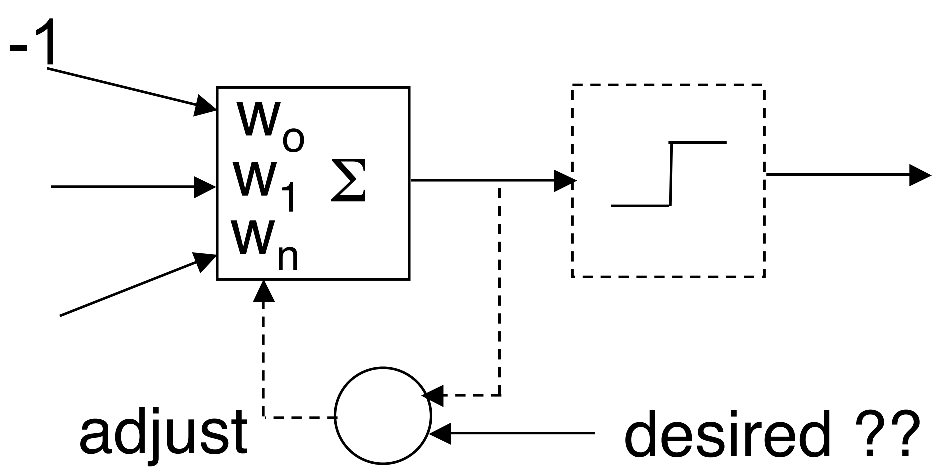
- With or without the threshold, the *Adaline* is **trained** based on the output of the *linear* function rather than the final output.



Adaline Training (2)

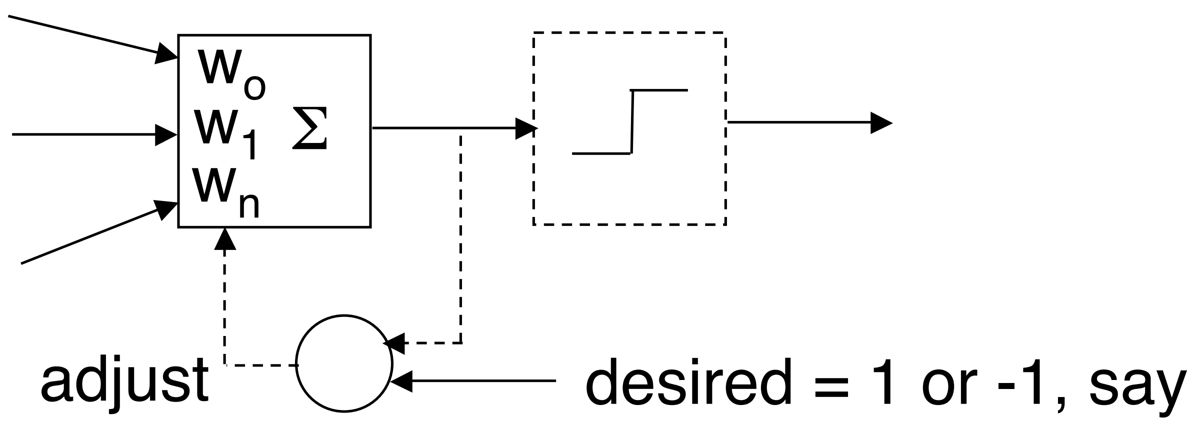
- The catch here is that we have to state the **desired** value in terms of the output of the linear part, rather than the output after the limiter.

- What is this for a classifier?



Adaline Training (2)

- A reasonable approach is to use any *nominal* target value such as -1 as desired for a "no" classification and a +1 for a "yes" classification.



Adaline Training (3)

- The formula for Adaline weight updating will be seen as very similar to the Perceptron:
  Add to the weights Δw where

$$\Delta w = \varepsilon\, \eta\, [\, -1, x_1,\ x_2\,,\ \ldots, x_n\,]$$

Adaline learning rule

only now the ***error* is not limited to 1, -1, 0 as before**; it can have a fractional value, since it is based on the output of the linear part of the device.

Adaline Training (4)

- One major difference from this vs. the Perceptron is that the learning rate can't be so arbitrary. It will generally need to be a lot less than 1.

- There is a theory that tells us how large we can make the learning rate.

Adaline Convergence (2)

- The Adaline admits a more refined stopping criterion:
  The **Mean-Squared Error (MSE)** is the average of the squares of the error taken over all samples. Squaring makes the measure insensitive to the sign of the error. It also provides certain analytic properties.

- This quantity ideally converges toward a specific minimum (which might never be exactly attained). The algorithm can be set to stop when the MSE reaches a desired value.

Adaline Example

- We'll use the same example as before. But now we'll train on the output of the linear portion and target for +1 for a "yes" answer and -1 for a "no" answer.
  - (4, 5)   +1
  - (6, 1)   +1
  - (4, 1)   -1
  - (1, 2)   -1
- Try a learning rate of 0.01.

Adaline Training Example

- Start with initial weights all 0.
- (In general, can use random weights.)
- Progress:
  - trying sample desired: 1, inputs: -1 4 5 output: -1
  - diff is 2
  - error is 1
  - new weights: -0.01 0.04 0.05

Adaline Training Example

- weights: -0.01 0.04 0.05
- trying sample desired: 1, inputs: -1 6 1 output: 1
- diff is 0
- error is 0.7
- new weights: -0.017 0.082 0.057

- trying sample desired: -1, inputs: -1 4 1 output: 1
- diff is -2
- error is -1.402
- new weights: -0.00298 0.02592 0.04298

Adaline Training Example

- trying sample desired: -1, inputs: -1 4 1 output: 1
- diff is -2
- error is -1.402
- new weights: -0.00298 0.02592 0.04298

- trying sample desired: -1, inputs: -1 1 2 output: 1
- diff is -2
- error is -1.11486
- new weights: 0.0081686 0.0147714 0.0206828

- **epoch 1**: wrong = 3, mse = 1.17463

Adaline Training Example

- epoch 1: wrong = 3, mse = 1.17463
- epoch 2: wrong = 2, mse = 1.11176
- epoch 3: wrong = 2, mse = 1.08817
- epoch 4: wrong = 2, mse = 1.07521
- epoch 5: wrong = 2, mse = 1.06563

  . . .
- epoch 30: wrong = 2, mse = 0.888344
- epoch 31: wrong = 1, mse = 0.881999

  . . .
- epoch 197: wrong = 1, mse = 0.363726
- epoch 198: wrong = 0, mse = 0.362493

- Final weights: 1.54436 0.273645 0.252003

Adaline Training Example

- Final weights: 1.54436 0.273645 0.252003

input	desired	weighted sum	actual
(4, 5)	+1	0.810235	1
(6, 1)	+1	0.359513	1
(4, 1)	-1	-0.197777	-1
(1, 2)	-1	-0.766709	-1

Alternate Rule Names

- Because the Adaline rule minimizes MSE, it is sometimes called the "**LMS rule**" [LMS = "least mean square"].
- The term "**Delta rule**" is also sometimes used, although this will be seen to be a rule for a more general class of networks.
- The term "**Widrow-Hoff**" rule is also used.

What function is minimized?

- Think of the inputs as being **constants**.
- The **weights** are the variables.
- We want to find the weights that minimize the **MSE** as a function of the weights.
- The fact that the MSE is defined analytically is a big help.

Gradient Descent

- Gradient descent is a method for finding the minimum of the error function (a function of the weights).

- It consists of computing the **gradient** of the function, then taking a small **step** in the **direction of negative gradient**, which hopefully corresponds to decreased function value, then repeating for the new value of the dependent variable.

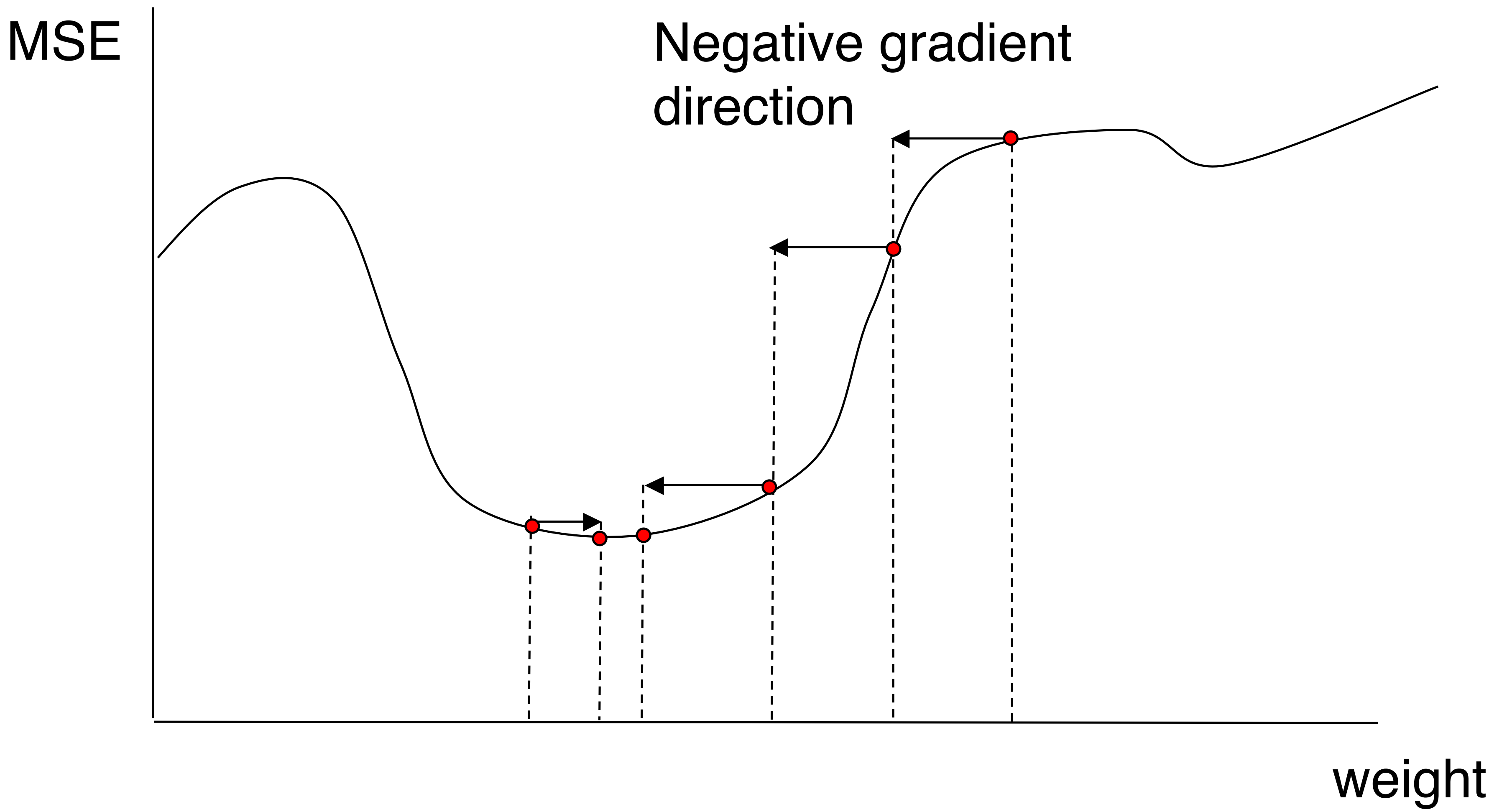
Gradient?

- The gradient is simply a generalization of the ordinary derivative d/dw to n dimensions.
- Specifically, it is the **vector** of partial derivatives, one component for each dimension:

$$\nabla_w = [\partial/\partial w_1 , \partial/\partial w_2, \ldots, \partial/\partial w_n]$$

Apply this operator to the error function.

Gradient Descent

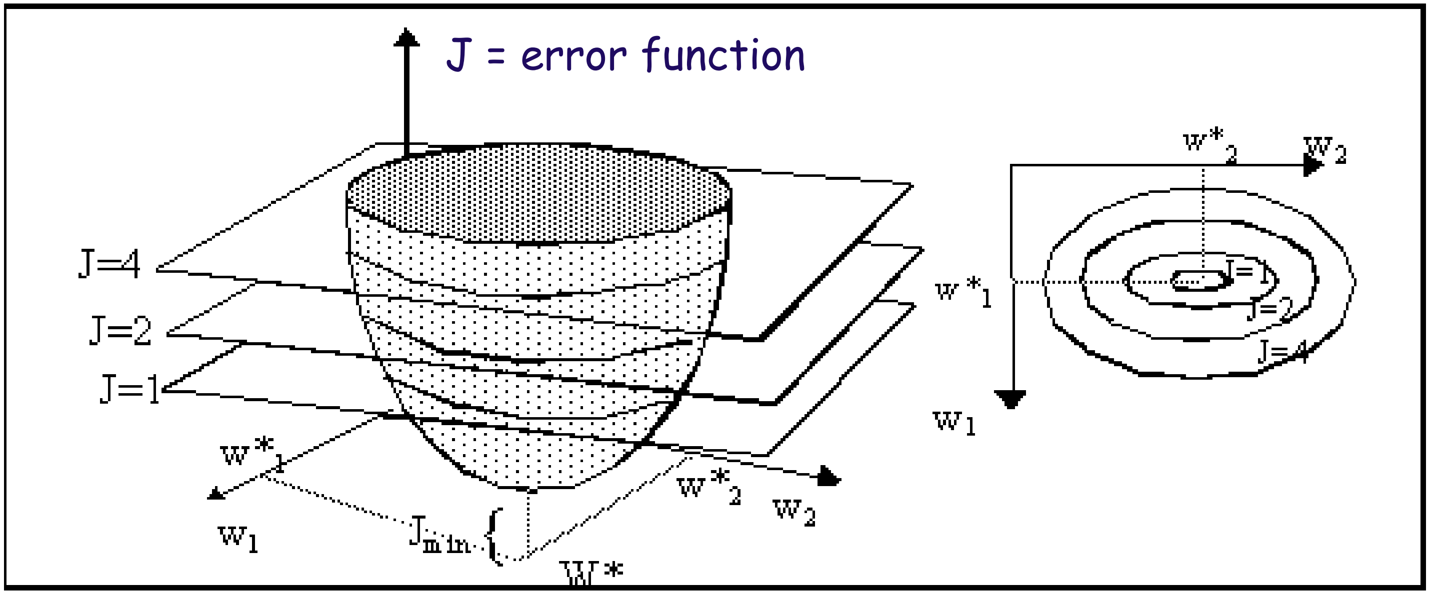


Gradient Descent

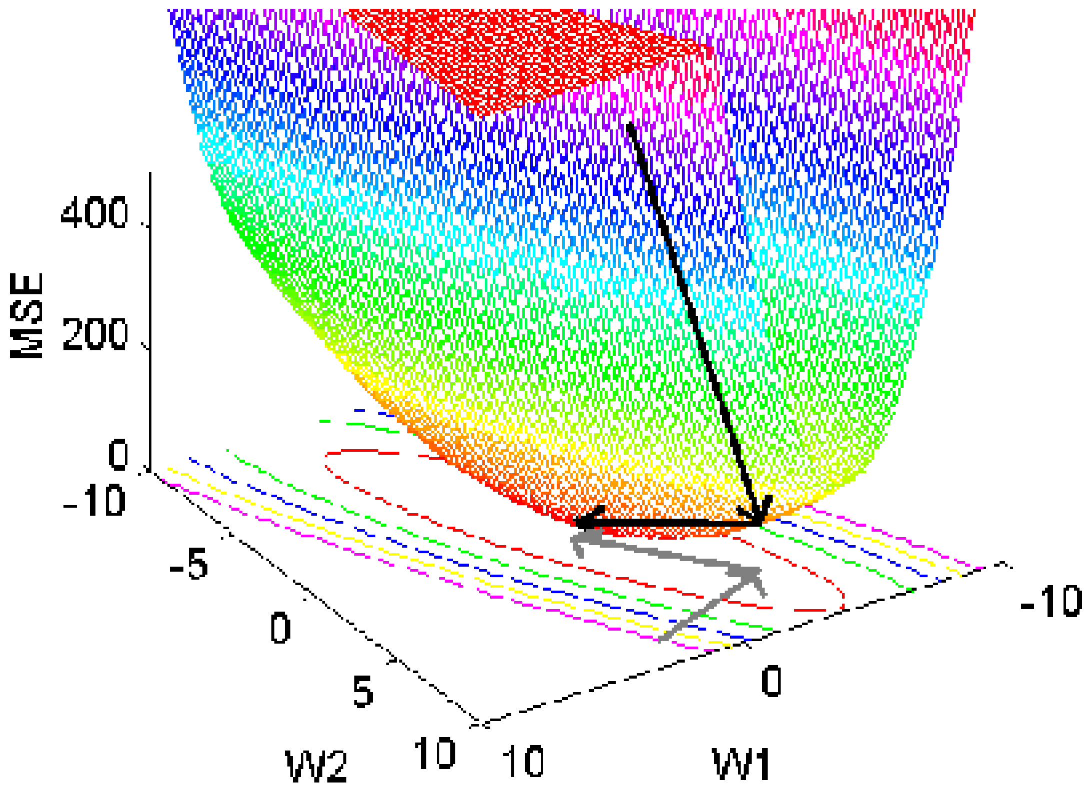
- The previous diagram is mainly to enhance our intuition.
- A single dimension for weights (including threshold or bias) is not typical. We need a two dimensional domain for just a single weight plus a bias.
- For the general case, the gradient is a **vector** of gradient components, one for each weight (including bias).

Error vs. 2-D Weight Space

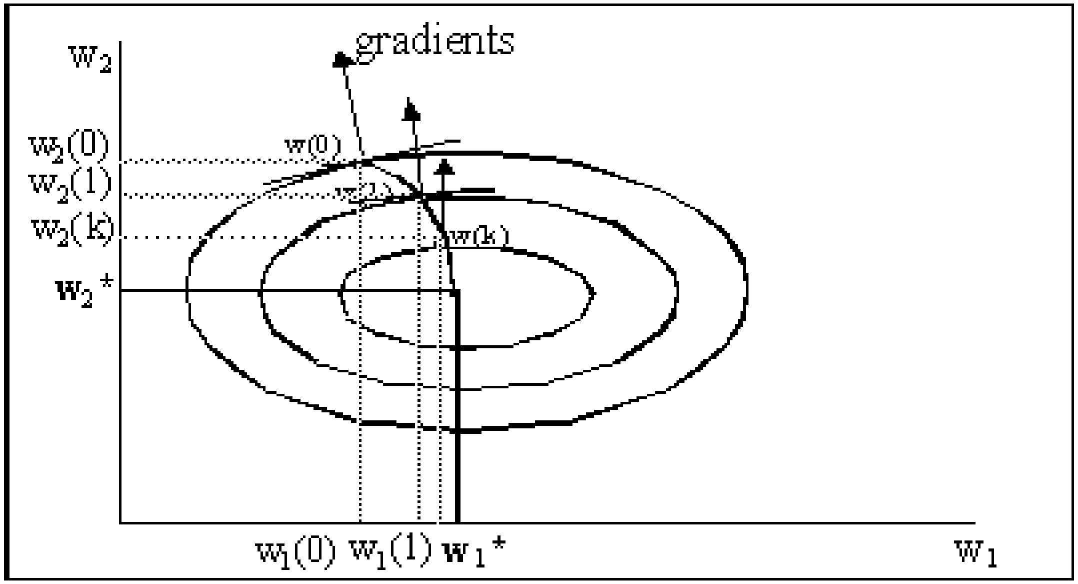
(one weight is threshold or bias)



2-D Gradient Descent



2-D Gradient Descent Projection



Computing Gradients for the Adaline

Note: squared

- MSE = $J(w) = \Sigma(\text{desired-actual})^2/n$
  where Σ is over n samples.
- 'desired' is a fixed value for each sample.
- actual = $\Sigma w_j x_j$ (Σ over input lines, including phantom input for threshold or bias)
- So $J(w) = \Sigma(\text{desired}- \Sigma w_j x_j)^2/n$

On-Line Approximation to Gradient

- "On-line" means based on a **single sample**, vs. "batch", which means using all samples

(note: no outer sum)

- $J \approx (d - \Sigma w_j x_j)^2$ (d = desired)
- i^{th} gradient component = $\partial J / \partial w_i$
  $= \partial / \partial w_i (d - \Sigma w_j x_j)^2$
  $= 2 \underbrace{(d - \Sigma w_j x_j)}_{= \text{error}, \, \varepsilon} \underbrace{\partial / \partial w_i (d - \Sigma w_j x_j)}_{-x_i} = -2 \varepsilon x_i$

Computing Gradients

- i^{th} gradient component = $-2\ \varepsilon\ x_i$
- However we want to move in the *direction* of ***negative*** gradient, tempered by the **learning rate** η, so:

  Amount to *add* to weight is

  $$\boxed{\Delta w_i = 2\ \varepsilon\ \eta\ x_i}$$

  which we recognize as the LMS (Adaline) rule (2 could be folded into η).

- This is the rule used in our earlier demonstration.

Vector Analysis of Gradient Descent for Adaline

- $MSE = J(w) = E[(d - w^T x)^2]$ (E = **expectation** or mean averaged over all samples)

$= E[d^2 - 2dw^T x + w^T x\, x^T w]$ (matrix-vector products)

$= E[d^2] - E[2dw^T x] + E[w^T x\, x^T w]$

$= E[d^2] - 2w^T E[dx] + w^T E[x\, x^T]\, w$

$= c - 2w^T h + w^T R\, w$, for appropriate const. c, h, R

Vector Analysis of Gradient Descent for Adaline

- $J(w) = c - 2w^T h + w^T R\, w$

  where $c = E[d^2]$, $h = E[d\, x]$, $R = E[x\, x^T]$

- This is a **quadratic form in w** with coefficients derived from the data vectors x.

- R is called the (auto-)**correlation matrix.**

Standard Quadratic Form in w

- $J(w) = c + w^T b + (1/2) w^T A\, w$

  where $c = E[d^2]$,
  $b = -2E[d\, x]$,
  $A = 2E[x\, x^T]$

- *A* is called the **Hessian** matrix. It is the matrix of **2nd partial derivatives** of the surface.

Analytic Minimization

- $\nabla_w J(w) = \nabla_w(c + w^T b + (1/2) w^T A\, w) =$
  $b + A\, w$

- It can be shown that if J has a **minimum**, it will be at a point w* where $\nabla J(w^*) = 0$,
  i.e. $b + A\, w^* = 0$
  i.e. $\boxed{w^* = A^{-1} b}$

  recalling $A = 2E[x\, x^T]$, $b = -2E[dx]$

Stable Points

- In general, $w^* = A^{-1}b$ is just a **stable** point.
- It may correspond to a minimum, maximum, or saddle.

Outline of Convergence Analysis of Gradient Descent for Adaline

- $\Delta w = 2\ \varepsilon\ \eta\ x$ (weight change at k^{th} step)
  $w(k+1) = w(k) + 2\ \eta\ \varepsilon(k)\ x(k)$

- Thus $E[w(k+1)] = E[w(k)] + 2\eta\ E[\varepsilon(k)\ x(k)]$
  … math …
  $= (I-2\eta R)\ E[w(k)] + 2\eta h$

  (I is the identity matrix, and h a constant vector)

- For convergence, the eigenvalues of the matrix $I-2\eta R$ must be **within the unit circle**.

Convergence of Gradient Descent for Adaline

- If λ_i is an eigenvalue of R, convergence requires $|1 - 2\, \eta\, \lambda_i| < 1$.
- This simplifies to $\eta < 1/\lambda_i$ for all eigenvalues λ_i, in particular for the maximum one.

$$\boxed{\eta < 1/\lambda_{max}}$$

Bound on learning rate for convergence of Adaline training by gradient descent

Worked Example
(from Neural Network Design)

Sample1 $\left\{ \mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{d}_1 = \begin{bmatrix} -1 \end{bmatrix} \right\}$ Sample 2 $\left\{ \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{d}_2 = \begin{bmatrix} 1 \end{bmatrix} \right\}$

Correlation Matrix: $\mathbf{R} = E[\mathbf{x}\mathbf{x}^T] = \frac{1}{2}\mathbf{x}_1\mathbf{x}_1^T + \frac{1}{2}\mathbf{x}_2\mathbf{x}_2^T$

$$\mathbf{R} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}\begin{bmatrix} -1 & 1 & -1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\lambda_1 = 1.0, \qquad \lambda_2 = 0.0, \qquad \lambda_3 = 2.0$$

$$\eta_{\max} = \frac{1}{\lambda_{max}} = \frac{1}{2.0} = 0.5$$

Training, 1st Epoch, with $\eta = 0.2 < \eta_{max}$

Sample 1 $$a(0) = \mathbf{W}(0)\mathbf{p}(0) = \mathbf{W}(0)\mathbf{x}_1 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = 0$$

$$\varepsilon(0) = d(0) - a(0) = d_1 - a(0) = -1 - 0 = -1$$

$$\mathbf{W}(1) = \mathbf{W}(0) + 2\eta\varepsilon(0)\mathbf{x}^T(0)$$

$$\mathbf{W}(1) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} + 2(0.2)(-1) \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}^T = \begin{bmatrix} 0.4 & -0.4 & 0.4 \end{bmatrix}$$

2nd Epoch

Sample 2 $a(1) = \mathbf{W}(1)\mathbf{p}(1) = \mathbf{W}(1)\mathbf{p}_2 = \begin{bmatrix} 0.4 & -0.4 & 0.4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = -0.4$

$$\varepsilon(1) = d(1) - a(1) = d_2 - a(1) = 1 - (-0.4) = 1.4$$

$$\mathbf{W}(2) = \begin{bmatrix} 0.4 & -0.4 & 0.4 \end{bmatrix} + 2(0.2)(1.4) \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}^T = \begin{bmatrix} 0.96 & 0.16 & -0.16 \end{bmatrix}$$

3rd Epoch

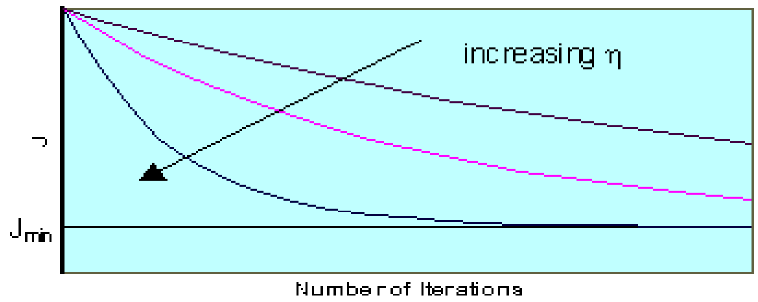
$$a(2) = \mathbf{W}(2)\mathbf{p}(2) = \mathbf{W}(2)\mathbf{p}_1 = \begin{bmatrix} 0.96 & 0.16 & -0.16 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = -0.64$$

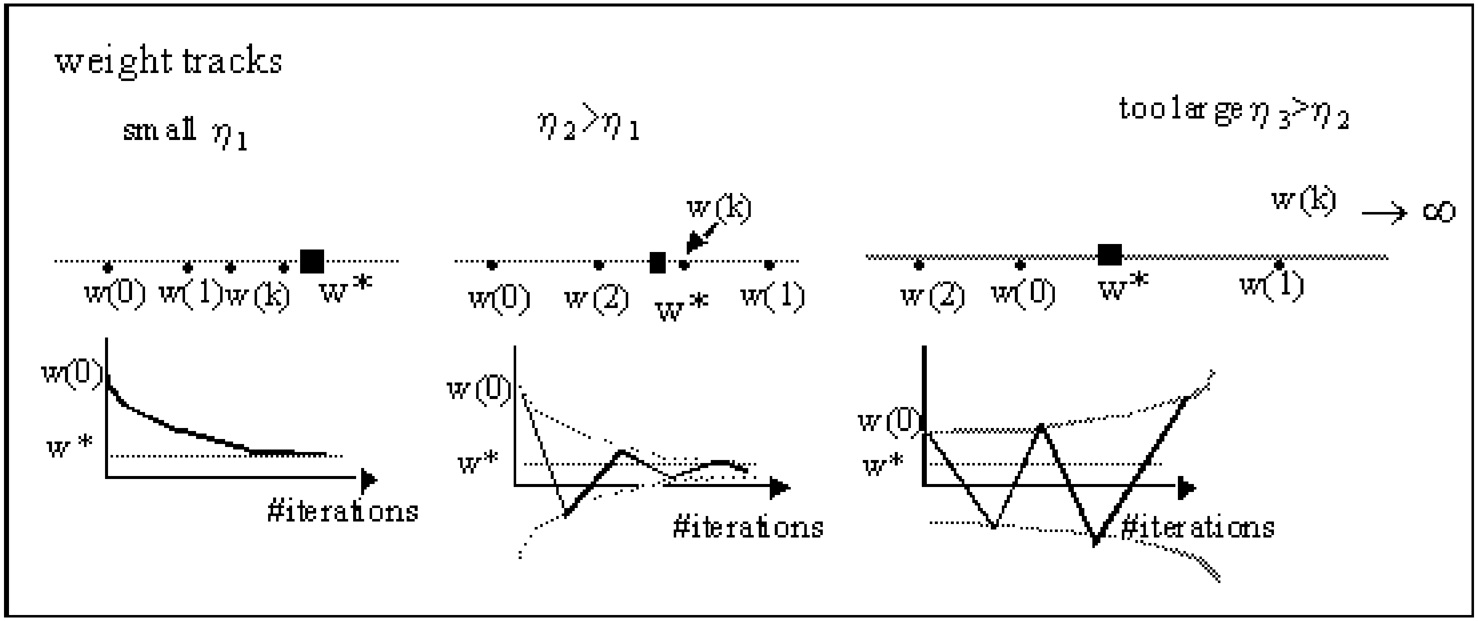
$$\varepsilon(2) = d(2) - a(2) = d_1 - a(2) = -1 - (-0.64) = -0.36$$

$$\mathbf{W}(3) = \mathbf{W}(2) + 2\eta\varepsilon(2)\mathbf{x}^T(2) = \begin{bmatrix} 1.1040 & 0.0160 & -0.0160 \end{bmatrix}$$

$$\mathbf{W}(\infty) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

"Learning Curve"



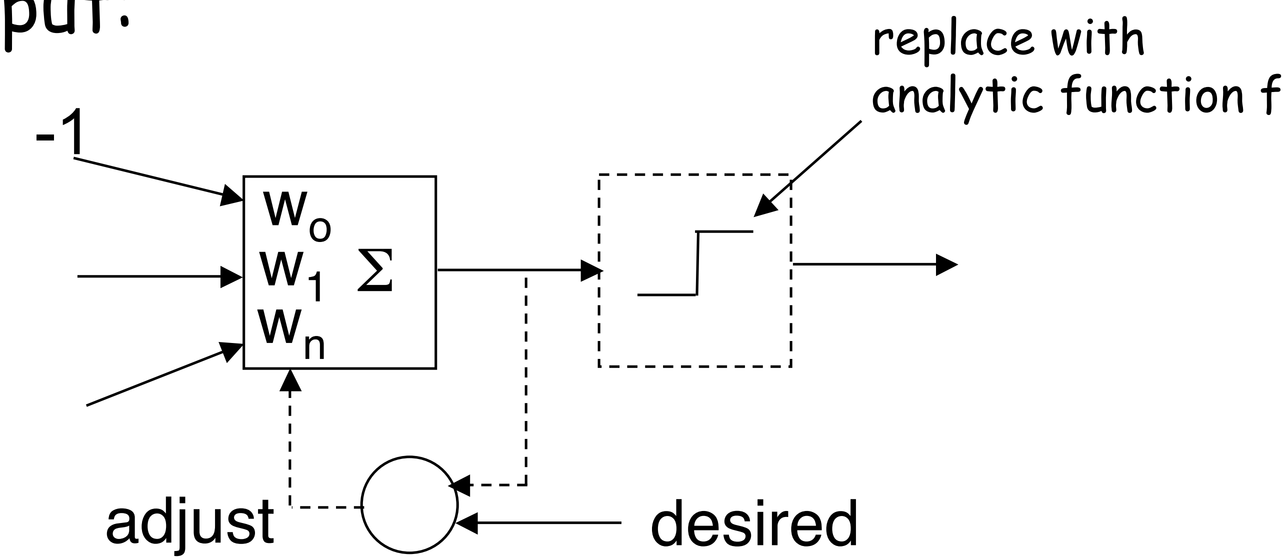


Generalizing the Adaline

- We have already discussed the problem of a threshold output in this picture

Generalizing the Adaline
(leading to multi-layer network training)

- Suppose that we replace the threshold stage with a general analytic function f and revert to expressing desired in terms of its output:



Generalizing the Adaline

- Consider again the derivation of the LMS rule:
  $J(w) = \Sigma(d - f(\Sigma w_j x_j))^2/n$

- $J \approx (d - f(\Sigma w_j x_j))^2$ (on-line approximation)

Generalizing the Adaline

- $J \approx (d - f(\Sigma w_j x_j))^2$ (on-line approximation)
- i^{th} gradient component $= \partial J / \partial w_i$

$= \partial / \partial w_i (d - f(\Sigma w_j x_j))^2$

$= 2\,(d - f(\Sigma w_j x_j))\, \partial / \partial w_i\, (d - f(\Sigma w_j x_j))$

$= -2\, \varepsilon\, \partial / \partial w_i\, f(\Sigma w_j x_j)$

$= -2\, \varepsilon\, f'(\Sigma w_j x_j)\, \partial / \partial w_i\, \Sigma w_j x_j$

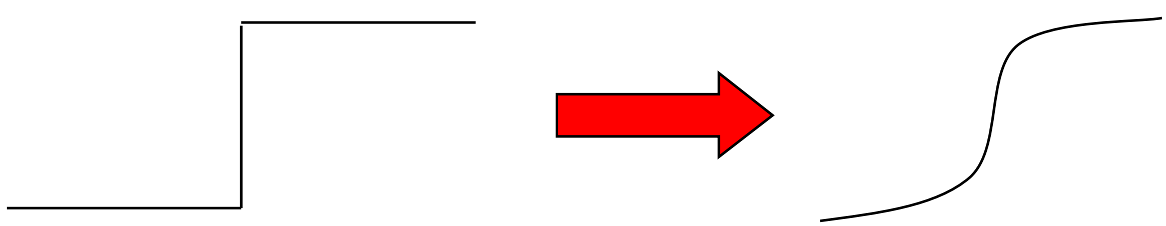
$= -2\, \varepsilon\, x_i f'(\Sigma w_j x_j)$ where f' is the ordinary derivative

Generalized LMS Rule (or Delta Rule)

- $\Delta w = 2\ \varepsilon\ \eta\ f'(\Sigma w_j x_j)\ x_i$ assuming that f has a derivative f'.
- $\Sigma w_j x_j$ is often called the **"net" value** or **"activation" value**, and f the **activation function**.
- For the special case of f being the **identity** function, this reduces to the LMS rule we had before.
- Next we see uses of the more general case.

Approximating Step Function Analytically

- In the Adaline with **threshold**, we can't very well treat the model analytically, due to the fact that we have a non-continuous function at the output.
- But we can *approximate* the non-continuous function with a continuous one:



Sigmoids

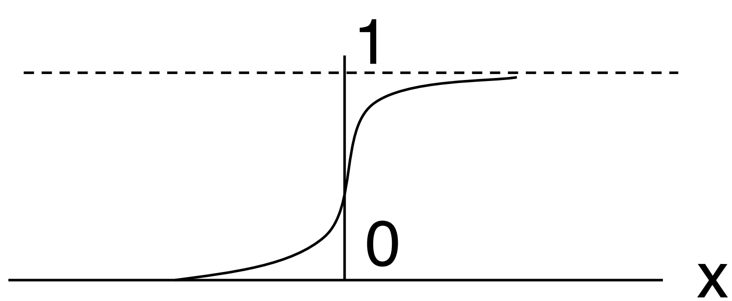
- The “S” shape on the right of the previous slide is called a sigmoid curve.
- This is a generic term and there are several different analytic functions that behave this way.

Logisitic Sigmoid

- Logisitic function ("logsig"–Matlab):

$$f(x) = 1/(1+\exp(-ax))$$

- $f'(x) = f(x)(1-f(x))$ shortcut to derivative

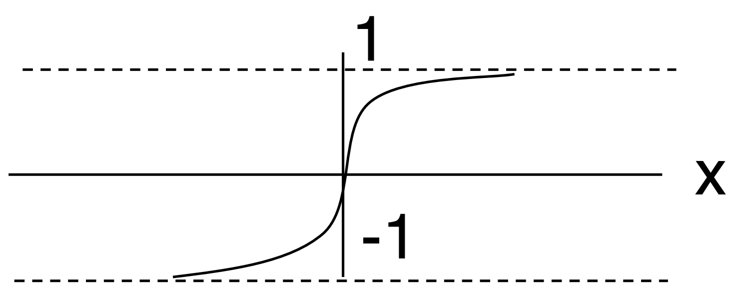


Hyperbolic Sigmoid

- Hyperbolic tangent function ("tansig"):

$$f(x) = \tanh(x) = (\exp(x)-\exp(-x))/(\exp(x) + \exp(-x))$$

- $f'(x) = 1-f^2(x)$ shortcut to derivative



Squashing Functions

- Sigmoids, step functions, and other functions that force their results to be in a limited range are called "squashing functions".
- It is generally accepted that biological neural system is based on such functions, as there are physical limits to the response level.