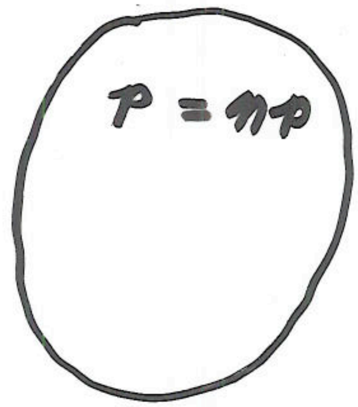
$\mathcal{P}$, $\mathcal{NP}$, and Beyond...

The world looks like...

or



(If $\mathcal{P} \neq \mathcal{NP}$) (If $\mathcal{P} = \mathcal{NP}$)

But actually it's even more interesting!

3SAT = { S | S is satisfiable }

$\overline{\text{3SAT}}$ = { S | S is not satisfiable }

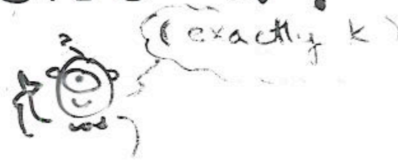
Oracles & The Polynomial Hierarchy !

(aka "Really hard problems, Really Really hard problems, Really Really Really hard problems...")

The Largest Clique Problem (LCP)

Input: Graph $G=(V,E)$ and parameter k.

Question: Does the largest clique in G have size k?



[Stockmeyer, 1976]

Minimum Equivalent Expression (MEE)

T, F, & variables connected with $\vee, \wedge, \neg$, and $\rightarrow$

Input: A well-formed boolean expression E & a parameter k.

Example

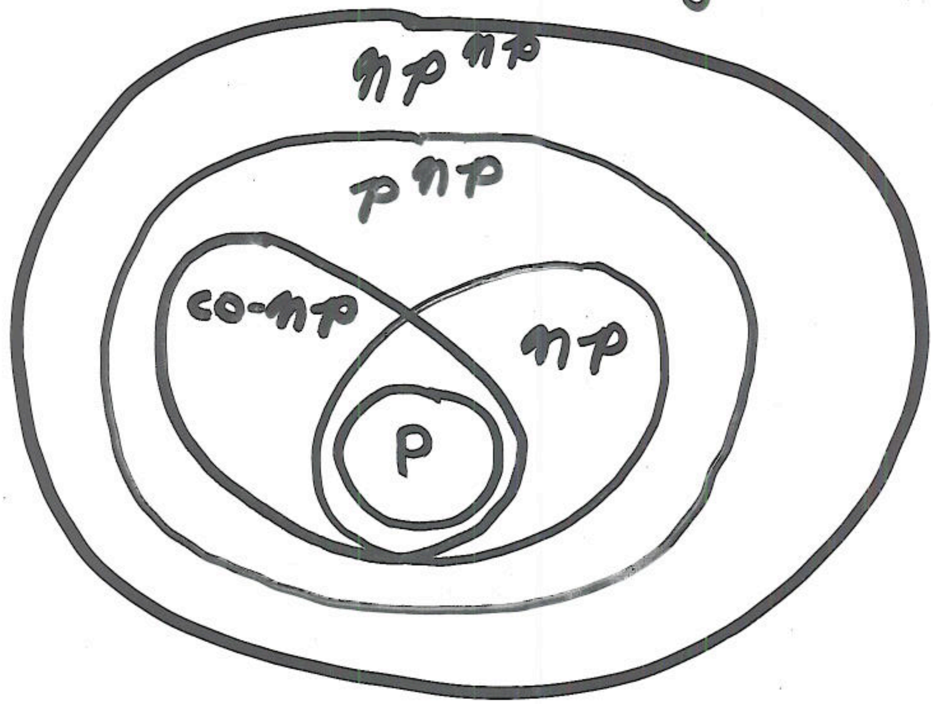
$x_1 \rightarrow (x_1 \vee \bar{x}_1)$

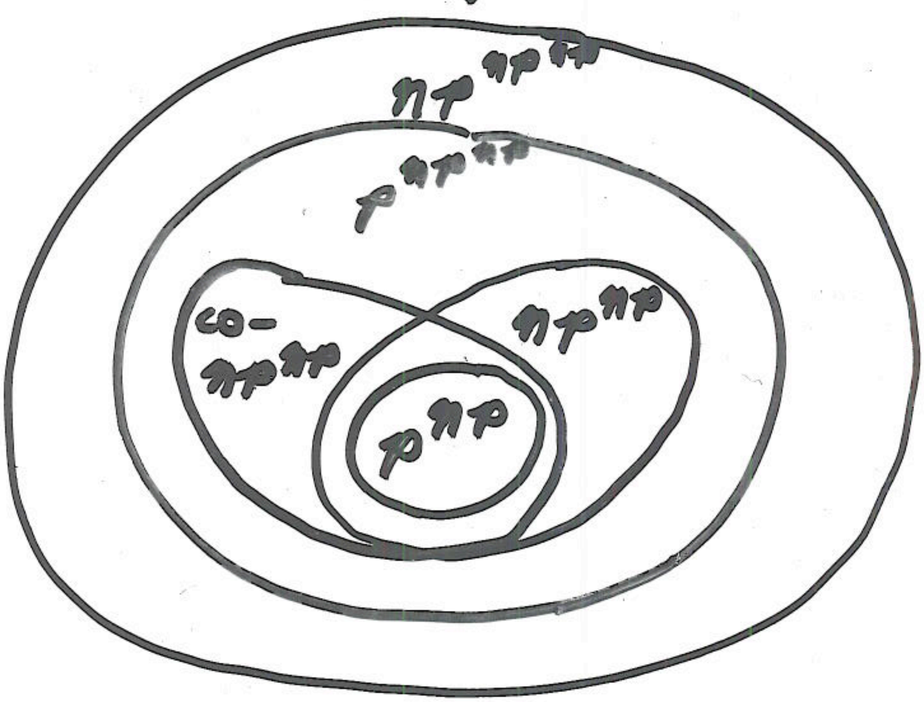
$(x_1 \vee x_2) \wedge (x_1 \vee \bar{x}_2)$

Question: Does there exist a well-formed boolean expression E' which contains at most k occurrences of literals s.t. E and E' are logically equivalent

A Revised Picture of the World...

The "Polynomial Hierarchy"





$\mathcal{P}$SPACE = { L / L is accepted by a deterministic TM (single tape) that uses at most $p(|w|)$ tape cells to decide w ($\in \Sigma^*$), for some polynomial $p(n)$. }

M always halts!

Claim 1: $\mathcal{P} \subseteq \mathcal{P}$SPACE

Claim 2: $\mathcal{NP} \subseteq \mathcal{P}$SPACE

Claim 3: co-$\mathcal{NP} \subseteq \mathcal{P}$SPACE

Claim 4: $\mathcal{P}^{\mathcal{NP}} \subseteq \mathcal{P}$SPACE

Claim 5: $\mathcal{NP}^{\mathcal{NP}} \subseteq \mathcal{P}$SPACE

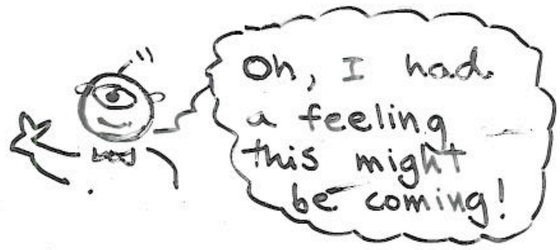
⋮

I claim that this game gets old fast!

PSPACE Completeness

Definition:

A language L ("decision problem") is PSPACE complete if



1.

2.

Lemma ("The significance of PSPACE completeness lemma.")

If L is PSPACE complete and $L \in P$ then...

Quick Quantification Questions

$$\exists x_1 \ \forall x_2 \ (x_1 \lor x_2)$$

$$\exists x_1 \ \forall x_2 \ (x_1 \land x_2)$$

$$\exists x_1 \ \forall x_2 \ (x_1 \land x_2) \lor (\overline{x}_1 \land \overline{x}_2)$$

$$\forall x_1 \ \exists x_2 \ (x_1 \land x_2) \lor (\overline{x}_1 \land \overline{x}_2)$$

PSPACE-complete Problems

Quantified Boolean Formula (QBF)

(Stockmeyer 1973)

Boolean exp. using $\wedge, \vee, \neg$

$$\exists x_1 \cdots \exists x_\ell \,\forall x_{\ell+1} \cdots \exists \cdots \forall x_m \; E(x_1, \ldots, x_n)$$

A Variant of QBF (Q3SAT)

$$\underbrace{\exists x_1 \forall x_2 \exists x_3 \forall x_4 \cdots \exists x_\ell}_{\text{quantifiers alternate between } \exists \text{ and } \forall} \;\; \underbrace{E'(x_1, \ldots, x_\ell)}_{\text{in 3SAT form}}$$

Geography

Instance:

C = { Albania, Algeria, Alabama, Afganistan, Liberia, Nepal, Norway, Yemen }

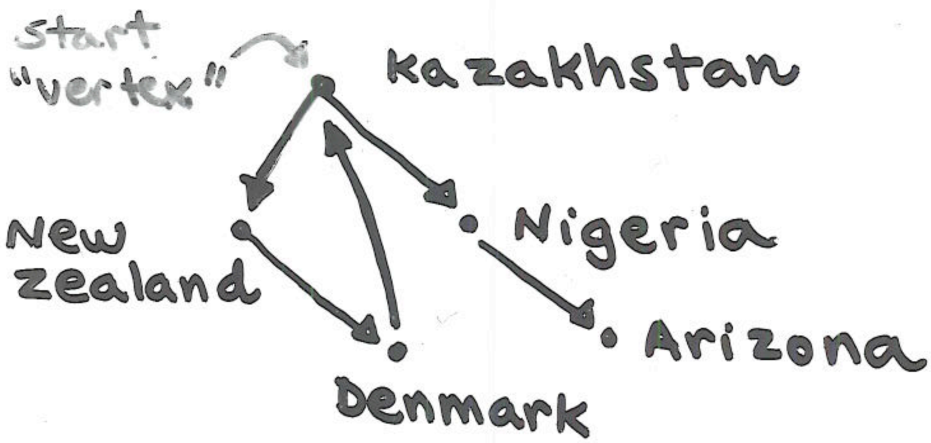
start position → Nepal

Question: Does there exist a winning strategy for player #1?

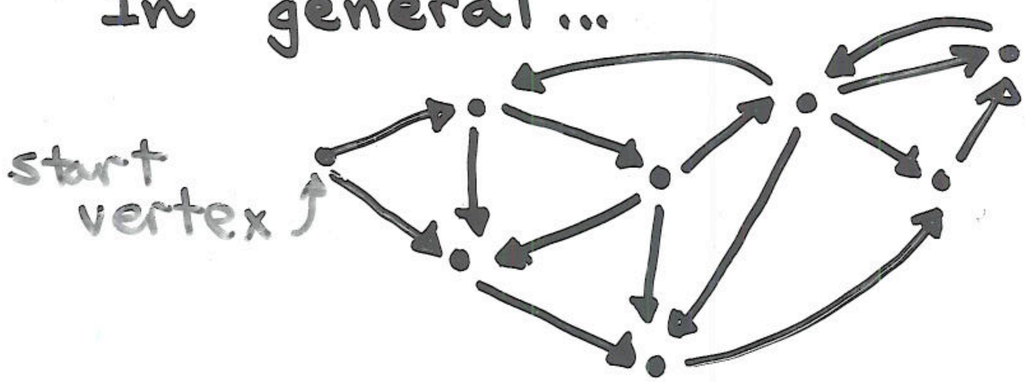
PSPACE-Completeness of Generalized Geography!

start country

{ Kazakhstan, Nigeria, New Zealand, Arizona, Denmark }



In general...

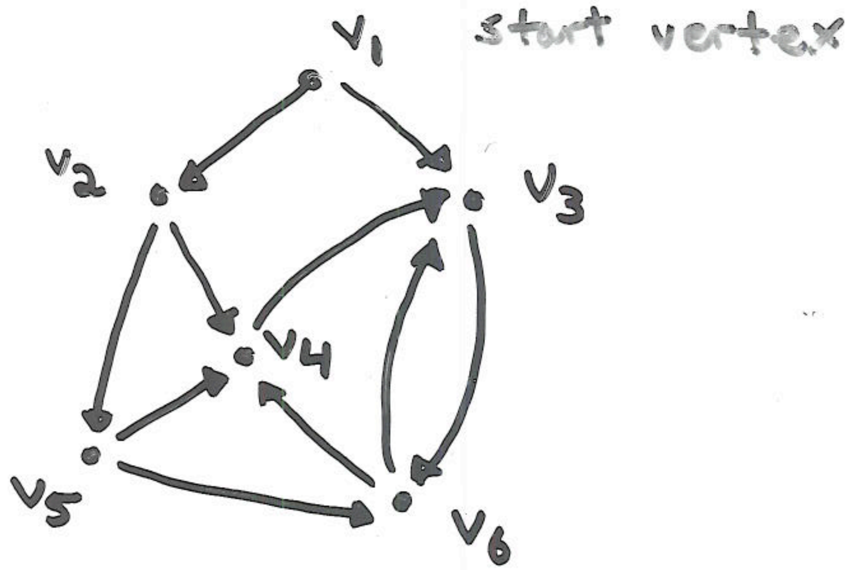


Claim: Generalized Geography is PSPACE-complete.

Proof:

1. GG $\in$ PSPACE

2. Q3SAT $\leq_p$ GG



2. Q3SAT $\leq_p$ GG

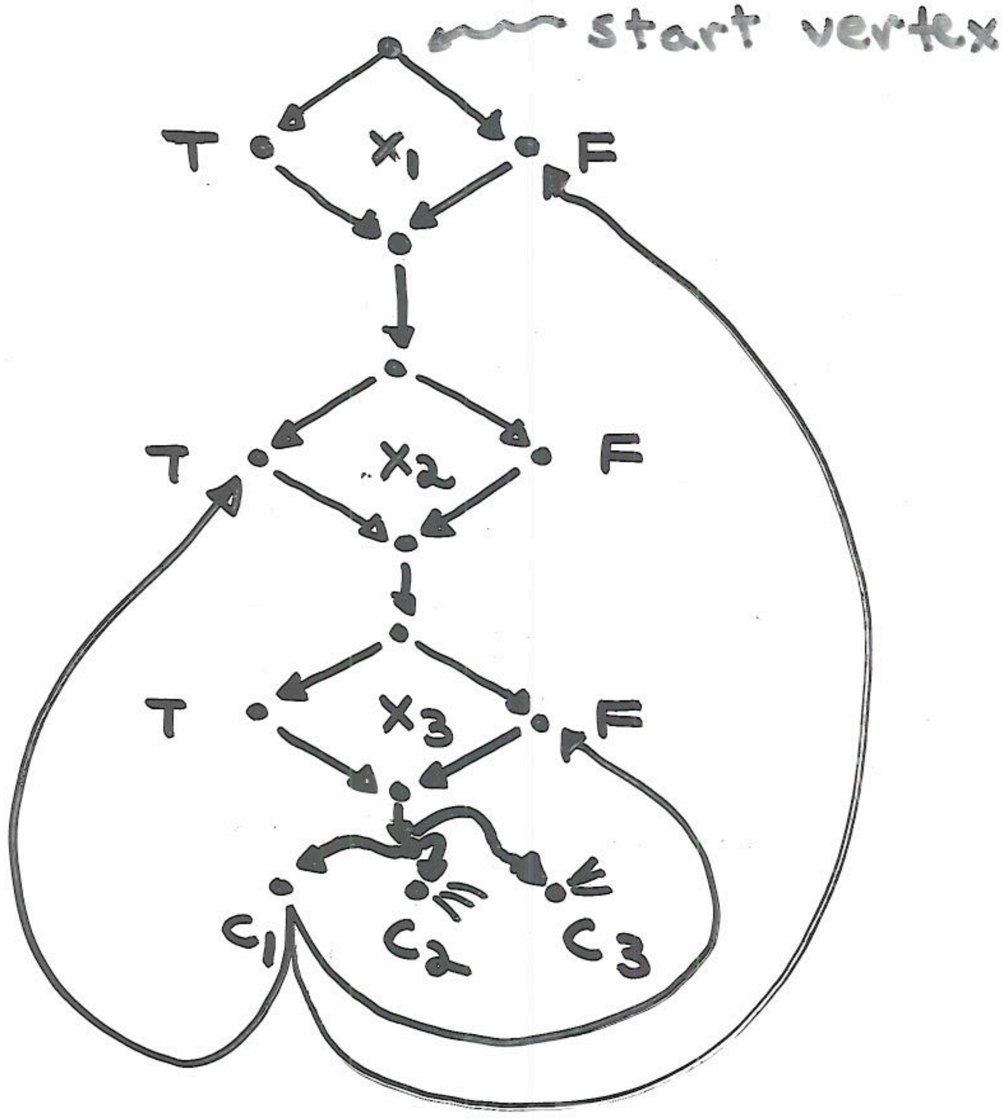
$$\exists x_1 \; \forall x_2 \; \exists x_3$$

C_2

C_1 $(x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3)$

C_3

poly-time $\Downarrow$



QBF is PSPACE-complete!

show...

1. QBF $\in$ PSPACE

2. $\forall L_i \in$ PSPACE

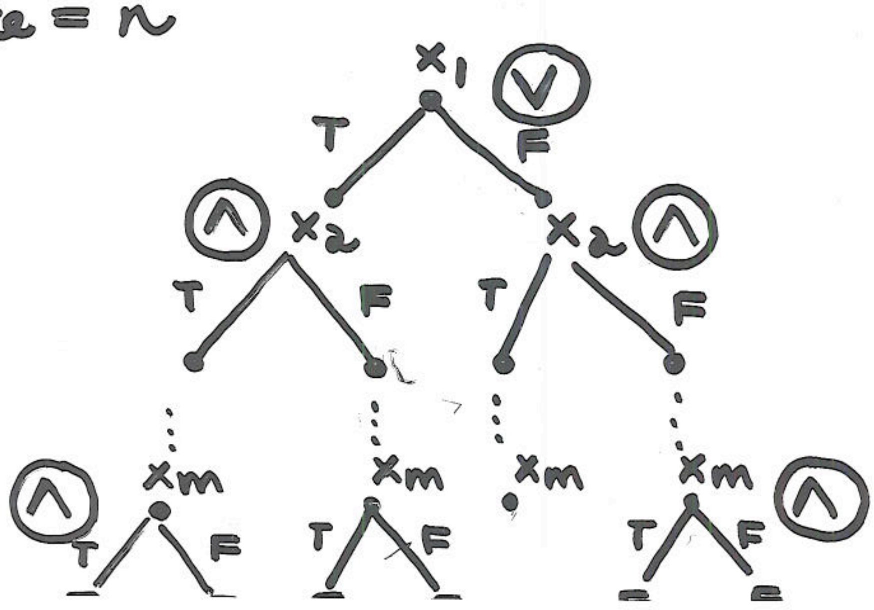
   $L_i \leq_p$ QBF

Claim: QBF $\in$ PSPACE

$$\exists x_1 \forall x_2 \forall x_3 \exists x_4 \cdots \forall x_m$$

$$\underbrace{(\bar{x}_1 \wedge x_2) \rightarrow ((x_3 \wedge x_4) \vee \bar{x}_5) \quad \cdots}_{E(x_1, x_2, \ldots, x_m)}$$

total length of problem instance = n



Let's try to prove that QBF is PSPACE complete using a "Cook-Levin-type" technique...

$w \xrightarrow{p(|w|)} w'$

Goal: $\forall\ L_i \in$ PSPACE, $L_i \leq_p$ QBF

Technique: If $L_i \in$ PSPACE $\exists$ a deterministic TM $M = (Q, \Sigma, \Gamma, \delta, q_{start}, q_{accept}, q_{reject})$ s.t. M decides words in $p(n)$ space.

M definitely always halts!

Introduce variables...

$H_{i,j}$ At time i, head of M is at position j on tape.

$Q_{i,j}$ At time i, state of M is q_j.

$S_{i,j,k}$ At time i, cell j contains character $\sigma_k \in \Gamma$.

Larry Stockmeyer's Ingenious "Trick"

- If $L_i \in \mathcal{P}SPACE$ then $\exists$ a deterministic poly-space TM $M = (\overset{\text{K states}}{Q}, \Sigma, \underbrace{\Gamma}_{\text{g symbols}}, \delta, \overset{\text{start}}{q_1}, \underbrace{q_2}_{\text{accept}}, \underbrace{q_3}_{\text{reject}})$ s.t. $\mathcal{L}(M) = L_i$.

  p(n)

- A configuration is a sequence of boolean variables:

  $H_1, \ldots, H_{p(n)}$, head position

  $Q_1, \ldots, Q_K$, state

  $S_{i,j}$ $\quad 1 \leq \overset{\text{position}}{i} \leq p(n)$

  $\quad 1 \leq \underbrace{j}_{\text{symbol index from } \Gamma} \leq g$

$$w = \sigma_{i_1} \sigma_{i_2} \sigma_{i_3} \cdots \sigma_{i_n}$$

$$\text{INIT}(C) =$$

$$(H_1) \wedge (Q_1) \wedge$$

$$(S_{1,i_1}) \wedge (S_{2,i_2}) \wedge \ldots \wedge (S_{n,i_n})$$

$$\wedge (S_{n+1,1}) \wedge \ldots \wedge (S_{p(n),1})$$

blank index

$$\text{ACCEPT}(C) = Q_2$$

$$\mathrm{LEGIT}(C) =$$

$$(H_1 \vee H_2 \vee \dots \vee H_{p(n)}) \wedge$$

$$(\overline{H_i} \vee \overline{H_j}) \wedge$$

all $i \neq j$

indep. i and j

$$(Q_1 \vee Q_2 \vee \dots \vee Q_K) \wedge$$

$$(\overline{Q_i} \vee \overline{Q_j}) \wedge$$

all $i \neq j$

position

symbol

$$(S_{1,1} \vee S_{1,2} \vee \dots \vee S_{1,g}) \wedge$$

$$(S_{2,1} \vee S_{2,2} \vee \dots \vee S_{2,g}) \wedge$$

$$\vdots$$

$$(S_{p(n),1} \vee S_{p(n),2} \vee \dots \vee S_{p(n),g}) \wedge$$

$$(\overline{S_{1,i}} \vee \overline{S_{1,j}}) \wedge$$

all $i \neq j$

$$\vdots$$

$$(\overline{S_{p(n),i}} \vee \overline{S_{p(n),j}})$$

We'll build $\alpha_i(C_1, C_2)$ Recursively...

$H_1, H_2, \ldots, H_{p(n)}, Q_1, Q_2, \ldots, Q_K, S_{1,1}, S_{i,j}$ $1 \le j \le$

$1 \le i \le p(n)$

$$\alpha_0(C_1, C_2) =$$

$\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_{p(n)}, \mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_K,$
$\mathcal{S}_{1,1}, \mathcal{S}_{i,j}$ $1 \le j \le g$

$1 \le i \le p(n)$

either $C_1 = C_2$...

How big is this?

$$[(H_1 \wedge \mathcal{H}_1) \vee (\overline{H_1} \wedge \overline{\mathcal{H}_1})] \wedge$$
$$[(H_2 \wedge \mathcal{H}_2) \vee (\overline{H_2} \wedge \overline{\mathcal{H}_2})] \wedge \ldots$$
$$\ldots \wedge [(S_{p(n),g} \wedge \mathcal{S}_{p(n),g}) \vee (\overline{S_{p(n),g}} \wedge \overline{\mathcal{S}_{p(n),g}})]$$

... or $C_1 \xrightarrow{1} C_2$...

deterministic!!! ☺

for $\delta(q_c, \sigma_d) = (q_e, \sigma_f, R)$

$$\wedge \; [(H_\ell \wedge Q_c \wedge S_{\ell,d}) \rightarrow (\mathcal{Q}_e \wedge \mathcal{S}_{\ell,f} \wedge \mathcal{H}_{\ell+1})]$$

one for every $1 \le \ell \le p(n)$

$\wedge$ all symbols in C_1 and C_2 match everywhere else

How big is this?

for $j \geq 1$

$$\alpha_j(C_1, C_2) = \exists C' \left[\text{LEGIT}(C') \wedge \alpha_{j-1}(C_1, C') \wedge \alpha_{j-1}(C', C_2) \right]$$