Assignment 11: (Un)decidability

Due: 1:15pm, Tuesday, November 23

- Emails about this assignment should be directed to `cs81help@cs.hmc.edu`.
- Grutor office hours in Platt are Sundays **7–10pm** and Mondays 9–11pm; Prof. Stone's office hours in Olin 1251 are MW 4–5pm and TR 3–4pm and by appointment.
- The usual collaboration rules apply. You may *discuss* an exercise with any other student(s) currently taking CS 81 as long as:
  - You contribute equally;
  - You come away from this discussion only with *understanding in your head* — no written materials or computer notes may be retained;
  - Your submission is authored solely by you, on a separate occasion.
- You should refer only to materials from this semester of CS 81 (lecture notes, handouts, textbooks, grutors, profs, etc.).
- Bring a writeup/printout to class on the due date. Illegible answers will get no credit.
- Make sure your submission includes your name!

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1. Read Chapter 4 (pages 165–182) of Sipser. Come up with (at least) two questions about the reading where you're not sure of the answer. These may relate to points where the book is confusing, or simply to some related question or conjecture that occurs to you while doing the reading.

2. What is the smallest alphabet Σ such that all Turing machines can be encoded as members of Σ^*? Justify your answer.

3. Consider the language of Turing machine descriptions

$$L = \{\langle M \rangle \mid M \text{ accepts at least 999 strings}\}.$$

Is L decidable, recognizable, or neither? Prove your answer.

4. Consider the language of Turing machine descriptions

$$L = \{\langle M, w \rangle \mid M \text{ accepts } w \text{ using at most 999 tape cells}\}.$$

Show that L is decidable.

5. For each of the following languages L, state whether L is decidable, recognizable, or neither. Carefully justify each answer.

   (a) $\{a\}$

   (b) $\{\langle M\rangle \mid L(M) = \{a\}\}$

   (c) $\{\langle M\rangle \mid M$ is the only TM accepting $L(M)\}$

   (d) $\{\langle M\rangle \mid M$ does not accept any string ending in 0 $\}$

   (e) $\{\langle M\rangle \mid L(M)$ is regular $\}$

   (f) $\{\langle M\rangle \mid M$ halts on ϵ in no more than 1000 steps $\}$

6. Goldbach's Conjecture states that every even integer greater than 2 can be expressed as the sum of two primes. It has not yet been proved, but (according to Wikipedia) has been verified for all even numbers up to 1.609×10^{18}. Explain how, given a (sadly, mythical) Halt checker, we could determine the truth of Goldbach's Conjecture once and for all.

7. **The Elusiveness of Undecidability.** It is widely known that Halting (membership in $H = \{\langle M, w\rangle \mid M$ halts on input $w\}$) is not decidable. But:

   (a) Consider the following machine, which we will call TM_1. It first confirms that the tape contains a number expressed in binary (and if not, rejects). It then enters the following loop:

   - All digits are zero, it halts and accepts (ending the loop).
   - Otherwise, it decrements the binary number by 1 (by completely overwriting it.)

   Does the machine TM_1 halt on the input 10000000000000?

   (b) Consider a similar machine, which we will call TM_2. It is exactly the same as TM_1, except that each time around the loop it *increments* the binary number by 1. Does the machine TM_2 halt on the input 10000000000000?

   (c) Why don't your answers to Parts (a) and (b) contradict the undecidability of Halting?

   (d) TM_1 and TM_2 are not very interesting. You are probably wondering, since Halting is undecidable, where the hard Turing Machines are.

   Suppose I were to design a very complicated TM_3, and prove mathematically that there is no method to determine whether TM_3 terminates on the input 10000000000000 (i.e., that you will never know). Show that this supposition leads to a contradiction.