Assignment 3: From Propositional Logic to Predicate Logic

Due: 1:15pm, Tuesday, September 14

- Emails about this assignment should be directed to `cs81help@cs.hmc.edu` .
- The usual collaboration rules apply. You may *discuss* an exercise with any other student(s) currently taking CS 81 as long as:
  - You contribute equally;
  - You come away from this discussion only with *understanding in your head* — no written materials or computer notes may be retained;
  - Your submission is authored solely by you, on a separate occasion.
- You should refer only to materials from this semester of CS 81 (lecture notes, handouts, textbooks, grutors, profs, etc.).
- Bring a writeup/printout to class on the due date. Illegible answers will get no credit. (For this reason, the grutors like you to use LaTeX. But it's up to you.)
- Make sure your submission includes your name!

1 Proof or No Proof?

For each of the following, either show there is no proof (by finding a valuation/model that makes the assumptions true and the conclusion false), or provide a natural deduction proof.

1. Assumption: $\neg p \vee (q \to p)$. Conclusion: $\neg p \wedge q$.
2. Assumption: $p \to q$ and $\neg p \to q$. Conclusion: q.
3. Assumption: $p \to q \to r$. Conclusion: $q \to p \to r$.
4. Assumption: $p \to q \to r$. Conclusion: $p \to r \to q$.
5. Assumption: $(p \to q) \to r$. Conclusion: $p \to (q \to r)$.
6. Assumption: $p \to q$ and $s \to t$. Conclusion: $p \vee s \to q \wedge t$.
7. Assumption: $p \to (q \to r)$ and $p \to q$. Conclusion: $p \to r$.

2 Hilbert Systems

As mentioned in class, valuations and truth tables provide an standard way of defining objectively "correct" or "true" arguments making conclusions from assumptions: $\mathcal{S} \models A$ holds if A is true in any situation in which the assumptions in the set $\mathcal{S}$ are all true.

We can then ask whether our "proof rules" guarantee that anything we prove is "correct" according to this standard. The argument sketched in class (and explained in more detail in Sections 1.4.3 and 1.4.4 of Huth and Ryan) was that our natural deduction rules gave us all and only "correct" arguments, i.e., that $\mathcal{S} \models A$ if and only if we can prove $S \vdash A$. Thus, the formulas that we can prove with no extra assumptions are exactly the tautologies.

Of course, we could ask whether *different* sets of rules yield the same set of "true" statements (tautologies). An alternative to natural-deduction systems are Hilbert systems.

The system we consider for this homework has the following three axiom schemes:

H1 $\vdash A \to (B \to A)$
H2 $\vdash (A \to (B \to C)) \to (A \to B) \to (A \to C)$
H3 $\vdash ((A \to \bot) \to \bot) \to A$.

That is, we have an infinite set of axioms. Each results from taking one of these three "patterns" and replacing A, B, and C by specific formulas (e.g., p, or $(p \to (q \to \bot))$).

There is one rule of inference, modus ponens:

If $\vdash A$ and $\vdash A \to B$
then $\vdash B$.

1. Show how to prove that $\vdash p \to p$. (Hint: use H2 once and H1 twice.)
2. The axiom schemes only tell us about $\to$ and $\bot$. Fortunately, all other logical connectives can be expressed using $\to$ and $\bot$. For example, $\neg A$ can be encoded as $A \to \bot$. Show how to express $A \wedge B$, $A \vee B$, $A \leftrightarrow B$, and $\top$ in terms of $\to$ and $\bot$.

3 Sequent Calculus

Hilbert Systems are compact, but hard to use. When people make arguments, their reasoning is closer to natural deduction. When computers search for proofs, they often use yet another way of organizing proof rules, known as *Sequent Calculus*.[1]

Attached is a short description of Sequent Calculus. As you might expect, $\vdash A$ is provable using the rules in Figure 7.1[2] if and only if A is a tautology.

When reading the handout, the following comments may be helpful:

[1] The word "calculus" means a formal system of rules, and has nothing to do with derivatives or integration.

[2] In fact, you don't even need the "Cut" rule, although it can be used to make very long proofs shorter. The intuition is that the Cut rule permits you to prove a helper lemma φ and to use that lemma (perhaps repeatedly) to prove Σ, rather than having to reprove the lemma every time you need it.

- The assumption is that $\neg A$ is always encoded as $A \to \bot$;
- Note that Γ, Δ, and Σ represent finite sequences (lists) of logical formulas. The notation Γ, Δ means to append/concatenate the sequences Γ and Δ into a single sequence.
- $\Gamma \vdash \Delta$ corresponds intuitively to the statement that "if all the formulas in Γ are true, then *at least one* of the formulas in Δ are true".
- Sequences can be empty or have a single element. In particular a sequence with one formula $p \wedge \neg p$ can be viewed as the concatenation of two sequences Γ, Δ where Γ is empty and Δ is the sequence of length 1 containing $p \wedge \neg p$ as its only element. It can also be viewed as Γ, Δ where Γ contains $p \wedge \neg p$ and Δ is empty.
- Make sure you understand how the proofs in Examples 7.1.3 and 7.1.5 use the rules from Figure 7.1.
- Rather than "introduction" and "elimination" rules, we have "left rules" and "right rules" — what to do if the logical operator appears in our list of assumptions, and what to do if the logical operator appears in or list of things to possibly prove.

1. Show a sequent calculus proof tree for $\vdash p \vee \neg p$ (that is, $\vdash p \vee (p \to \bot)$).
2. Show a sequent calculus proof tree for $p \to q, p \to r, p \vdash q \wedge r$.
3. Show a sequent calculus proof tree for $\neg(p \wedge q) \vdash \neg p \vee \neg q$ (that is, ...).

(Hint: start from the root of the trees and work your way up.)

4 Thinking in Predicate Logic

Do Exercise 2.1.4(a-k), found on page 158 of Huth and Ryan.

Extra Credit [10%]

[This problem is hard; don't spend too much time on it unless you find it interesting.]

The formula $\neg\neg p \to p$ is a tautology, but it is not provable in constructive (a.k.a. intuitionistic) logic. That is, to prove it you need to a non-constructive rule like proof-by-contradiction, or the law of the excluded middle, or (for a trivial proof) $\neg\neg$-elimination.

However, a constructive logician will agree that the formula isn't false: $\neg\neg(\neg\neg p \to p)$ *is* provable constructively, without any non-constructive rules. Provide a constructive proof. (You will need $\neg$-introduction, but that is a rule *different* from proof-by-contradiction!)