Assignment 4: Predicate Logic
Due: 1:15pm, Tuesday, September 21

Make sure you understand Section 2.3 of Huth & Ryan and can follow the proofs there.

1 Formal Logic

Give natural deduction proofs of the following:

1. \(\exists x. (R(x) \rightarrow B(x)) \vdash (\forall x. R(x)) \rightarrow (\exists x. B(x))\)
   (e.g., \(R(x) = "x \ is \ real"\), \(B(x) = "x \ is \ bubbly"\))

2. \((\forall x. L(x)) \lor (\forall x. F(x)) \vdash \forall x. (L(x) \lor F(x))\)
   (e.g., \(L(x) = "x \ is \ lost"\), \(F(x) = "x \ is \ found"\))

3. \(\forall x. (I(x) \lor U(x)) \vdash (\forall x. I(x)) \lor (\exists x. U(x))\)
   (e.g., \(I(x) = "x \ is \ interesting"\), \(U(x) = "x \ is \ unremarkable"\))

4. \(\forall x. (p \rightarrow Q(x)) \vdash p \rightarrow \forall x. Q(x)\)
   (e.g., \(p = "I'm \ wearing \ earplugs"\), \(Q(x) = "x \ is \ quiet"\))

5. \(\vdash \exists x. (D(x) \rightarrow \forall y. D(y))\)
   (The drinker's paradox: there is at least one person satisfying "if he/she drinks beer, then everyone drinks beer.")

6. \(\neg(\exists x. A(x)) \lor (\forall x. V(x)), p \rightarrow \forall x. D(x) \vdash \forall y. \forall z. [(\neg A(z) \lor V(y)) \land (p \rightarrow D(y))]\)
   (e.g., \(A(x) = "x \ is \ an \ avocado"\), \(V(x) = "x \ is \ a \ vegetable"\), \(p = "I'm \ hungry"\), \(D(x) = "x \ is \ delicious"\))

7. \(\forall x. \neg S(x, x), \forall x. \forall y. \forall z. S(x, y) \land S(y, z) \rightarrow S(x, z) \vdash \forall x. \forall y. S(x, y) \rightarrow \neg S(y, x)\)
   (e.g., \(S(x, y) = "\text{movie } x \ is \ a \ sequel \ to \ movie \ y"\))
2 Informal Logic

Here is some background in set theory: for all sets \(X\) and \(Y\) and all elements \(z\),

\[
\begin{align*}
  z \in (X \cup Y) & \iff (z \in X) \lor (z \in Y) \\
  z \in (X \cap Y) & \iff (z \in X) \land (z \in Y) \\
  z \in (X \setminus Y) & \iff (z \in X) \land \neg (z \in Y) \\
  X \subseteq Y & \iff \forall x. (x \in X \rightarrow x \in Y) \\
  X = Y & \iff (X \subseteq Y) \land (Y \subseteq X) \\
  X \text{ meets } Y & \iff \exists z. z \in (X \cap Y)
\end{align*}
\]

For each of the following two propositions, give

(a) a proof in *mathematical English*. (For the first one, you only need to complete the proof provided.) Your proof should be 1–3 paragraphs of English text, as one might find in a math book or textbook.

The only mathematical symbols in your proof should be set-theory symbols (\(\in\), \(\notin\), \(\subseteq\), \(\cup\), etc.). Do not include any formal logic symbols (\(\forall\), \(\exists\), \(\land\), \(\neg\), etc.)

Your proof should be complete and without holes. They will probably not be great literature but, where possible, strive for clarity.

(b) Identify four natural deduction rules that correspond to (explicit or implicit) steps in your proof, and at least one part of each proof to which that rule corresponds.

1. \(A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)\).

   (a) **Proof.** Let \(x \in A \setminus (B \setminus C)\) be given. Then \(x \in A\) and \(x \notin B \setminus C\). That is, \(x \notin B\) or \(x \in C\). If \(x \notin B\), then \(x \in A \setminus B\); if \(x \in C\) then \(x \in A \cap C\). In either case, \(x \in (A \setminus B) \cup (A \cap C)\). Thus \(A \setminus (B \setminus C) \subseteq (A \setminus B) \cup (A \cap C)\).

   ... \(\Box\)

2. If \(A\) meets \(B\), then \(A \cup B \not\subseteq (A \setminus B) \cup (B \setminus A)\).