

## Assignment 5: Cardinality and Predicate Logic

Due: 1:15pm, Tuesday, September 28

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- Emails about this assignment should be directed to `cs81help@cs.hmc.edu`.
  - Grutor office hours in Platt are Sundays 8-10pm and Mondays 9-11pm; Prof. Stone's office hours in Olin 1251 are MW 4-5pm and TR 3-4pm and by appointment.
  - The usual collaboration rules apply. You may *discuss* an exercise with any other student(s) currently taking CS 81 as long as:
    - You contribute equally;
    - You come away from this discussion only with *understanding in your head* — no written materials or computer notes may be retained;
    - Your submission is authored solely by you, on a separate occasion.
  - You should refer only to materials from this semester of CS 81 (lecture notes, handouts, textbooks, grutors, profs, etc.).
  - Bring a writeup/printout to class on the due date. Illegible answers will get no credit.
  - Make sure your submission includes your name!
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**Make sure you have read and understood Section 2.4 of Huth & Ryan.**

### 1 Proof or No Proof?

For each of the following, give a natural deduction proof if one exists. If there is no proof, you should

1. Define a model that makes the assumptions true and the conclusion false;
2. Briefly explain why the assumptions are true and the conclusions are false;
3. *Also*, find an alternate model that makes the conclusion *true*. (It doesn't matter whether this new model makes the assumptions true or not; this is just more practice with models.)

1. Assumption:  $\forall x. \forall y. (S(x, y) \rightarrow S(y, x))$

Conclusion:  $\forall x. \neg S(x, x)$

2. Assumption:  $\top$  (i.e., no assumption)

Conclusion:  $\forall x. \forall y. (S(x, y) \rightarrow \exists w. (S(x, w) \wedge S(w, y)))$

3. Assumption:  $\forall x. ((P(x) \rightarrow Q(x)) \wedge (Q(x) \rightarrow P(x)))$   
Conclusion:  $(\forall x. P(x)) \rightarrow (\forall x. Q(x))$
4. Assumption:  $(\forall x. P(x)) \rightarrow (\forall x. Q(x))$   
Conclusion:  $\forall x. ((P(x) \rightarrow Q(x)) \wedge (Q(x) \rightarrow P(x)))$
5. Assumption:  $(\forall x. P(x)) \rightarrow q$   
Conclusion:  $\forall x. (P(x) \rightarrow q)$
6. Assumptions:  $\exists x.P(x), \quad \exists y.Q(y)$   
Conclusion:  $\exists z. (P(z) \wedge Q(z))$
7. Assumption:  $(\exists x.P(x)) \vee (\exists y.Q(y))$   
Conclusion:  $\exists z. (P(z) \vee Q(z))$
8. Assumption:  $\forall x. \exists y. S(x, y)$   
Conclusion:  $\exists y. \forall x. S(x, y)$

## 2 Cardinality

Carefully argue whether the following sets are countable or not.

1. The set of partial functions from  $\mathbb{N}$  to  $\mathbb{N}$  whose support is a finite set.

[A total function is one that, for any input, produces an output. A *partial* function is one that, for some or all particular inputs, always returns “don’t know” or “undefined” (often written  $\perp$  although it’s not a truth value) rather than returning an output. The *support* of a partial function (sometimes called the domain or domain of definition) is the collection of inputs that provide non- $\perp$  output.

For example, we could have a partial function  $\text{sqrt} : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$\text{sqrt}(x) := \begin{cases} \sqrt{x} & \text{if } x \geq 0; \\ \perp & \text{otherwise.} \end{cases}$$

The support of  $\text{sqrt}$  is then the set of all nonnegative real numbers.]

2. The set of (unlabeled) binary trees. (This was one of the sets defined inductively during the very first lecture of CS 81).
3. The set of flow networks with rational capacities (finite directed graphs, where each directed edge is labeled with a rational number called the “capacity” of that edge).
4. The set of Java programs that typecheck.

### 3 Bounded Quantifiers

In class, we mentioned that the “bounded quantifier” notation (where we restrict the individuals being quantified over) can be translated into the more primitive notions on the right.

$$\begin{aligned}\exists x \in S. P(x) &\longrightarrow \exists x. (x \in S \wedge P(x)) \\ \forall x \in S. P(x) &\longrightarrow \forall x. (x \in S \rightarrow P(x)) \\ \exists x \leq n. P(x) &\longrightarrow \exists x. (x \leq n \wedge P(x)) \\ \forall x \leq n. P(x) &\longrightarrow \forall x. (x \leq n \rightarrow P(x))\end{aligned}$$

Some people are surprised that the translations are different for the different quantifiers: bounded- $\exists$  becomes a logical-and, while the translation of bounded- $\forall$  becomes an implication.

1. Describe a model  $\mathcal{M}_1$  (where the set of individuals is the set of  $\mathbb{N}$  of natural numbers, and the relation  $\leq$  is interpreted as the actual less-than relation on  $\mathbb{N}$ ) that makes

$$\forall x \leq n. f(x) \leq m$$

and hence

$$\forall x. (x \leq n \rightarrow f(x) \leq m)$$

true but makes

$$\forall x. (x \leq n \wedge f(x) \leq m)$$

false. (That is, complete the model by giving interpretations of the function  $f$  and the constants  $n$  and  $m$ .)

2. Describe a model  $\mathcal{M}_2$  (where the set of individuals is  $\mathbb{N}$  and the relation  $\leq$  is interpreted as less-than on  $\mathbb{N}$ ) that makes

$$\exists x. (x \leq n \rightarrow f(x) \leq m)$$

true but makes

$$\exists x \leq n. f(x) \leq m$$

and hence

$$\exists x. (x \leq n \wedge f(x) \leq m)$$

false.