Applied Logics

Chris Stone

September 28, 2010

Outline

- How can there be more than one logic?
  Why would you want more than one?
- Constructive Logic
  - Research idea: RZ
- Linear Logic
- Temporal Logic
  - Research idea: CSL

Computable Mathematics

Traditional CS is closely connected to *discrete* mathematics

Data is finite (though arbitrarily large)

- Lists
- Trees
- Hash tables
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But what if you want to compute with

- Real numbers
- Infinite sequences and limits
- Integration and differentiation
- Arbitrary continuous functions
- Arbitrary open (or closed, or compact) subsets of the plane
- Differential equations

Problems with Floating-Point: Limited Range

Floating-point numbers are a *finite* subset of the *rationals*, i.e,

$$\pm(2^{24}+m)2^{n} \qquad \begin{cases} 0 \leq m < 2^{24} \\ -140 \leq n \leq 103 \end{cases}$$

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$$\begin{gathered}
340282346638528859811704183484516925440 \\
34028232\underline{6356119256160033759537265639424} \\
\vdots \\
1/2^{126} \\
+0 \\
-0 \\
1/2^{126} \\
\vdots \\
-340282326356119256160033759537265639424 \\
-340282346638528859811704183484516925440
\end{gathered}$$

Problems with Floating-Point: Approximate Answers

Round-off and cancellation matter!

```
//  1/1 + 1/2 + ... + 1/(N-1)
float sum1 = 0.0;
for (int n = 1; n < N; ++n)
   sum1 += 1.0 / float(n);

//  1/(N-1) + ... + 1/2 + 1/1
float sum2 = 0.0;
for (int n = N-1; n > 0; --n)
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When $\mathtt{N} = 10^8$, $\mathtt{sum1} = 15.4037$ and $\mathtt{sum2} = 18.8079$

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   - "Real" real numbers
   - "Real" complex numbers?
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   - Square root of a complex?
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3. Can we do these efficiently (at least in practice)?

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First idea: infinite sequence of decimal digits

$$\begin{aligned} 1/2 &= [5, 0, 0, 0, 0, 0, 0, \ldots] \\ 1/30 &= [0, 3, 3, 3, 3, 3, 3, 3, \ldots] \end{aligned}$$

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In practice, we'd produce digits *on demand*, e.g.,

$$\texttt{real} \quad \equiv \quad \texttt{nat} \rightarrow \{0,\ldots,9\}$$

Critique

The operation "Divide by 10" is computable! (How?)

$$1, 4, 1, 5, 9, 2, 6, 5, 3, 5, \ldots$$

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$$
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\hline
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Suppose you check 100 digits and see 0, 3, 3, 3, 3, . . .
What's the *first* digit of the output?

Better Representations of Reals

As just a few examples among many

- Infinite sequences of *signed* digits (e.g., -9 to 9)
- Rapidly converging Cauchy Sequences:

  $$\{\ a \in \mathbb{Q}^{\mathbb{N}} \mid \forall 0 \leq i < j.\ |a_i - a_j| < 2^i\ \}$$

  (even better: use dyadic rationals!)
- Converging infinite sequences of open intervals (with rational endpoints)

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These are the same results that one can get by constructive reasoning.

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- Compare:

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- Close connection between computable and constructive.

The RZ System

(Program developed with Andrej Bauer, University of Ljubljana)

- Input: Specifications in constructive logic
  - Sets, functions, predicates exist
  - Certain (constructive) axioms hold
- Output: Interfaces describing code
  - Representations must exist
  - Functions operating on these representations must exist
  - They must satisfy certain (classical!) properties.

Formal Basis: Realizability

Realizability dates back to Kleene, as an attempt to interpret constructive logic classically.

Embarrassingly short summary of a rich topic:

- A realizer of a mathematical object
  (e.g., set, vector, tuple, group, smooth manifold)
  is an implementation.
- A realizer of a mathematical proposition
  (e.g., $\forall n \in \mathbb{N}.\ \forall x \in \mathbb{R}.\ \exists y \in \mathbb{R}.\ |x - y^2| < \frac{1}{n}$)
  is its "constructive" content.

Realizers are tedious to describe by hand. Hence, RZ.

RZ Example: A Decidable Set

```
Parameter s : Set.
Axiom decide:
   forall a b : s, (a = b) \/ not (a = b).
```

```
type s
val decide : s -> s -> bool
     // forall a b : ||s||,
     //    if decide a b then
     //      a =s= b
     //    else
     //      not (a =s= b)
```

RZ Example: Finitely enumerable sets

The axioms (not shown) yield a data structure for finite "sets" supporting:

- The empty set
- A way to add an element to a set
- A "fold" or "reduce" operation, for binary operators that are
  - Commutative (`f(x,y) = f(y,x)`)
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- *Can* compute empty/non-empty

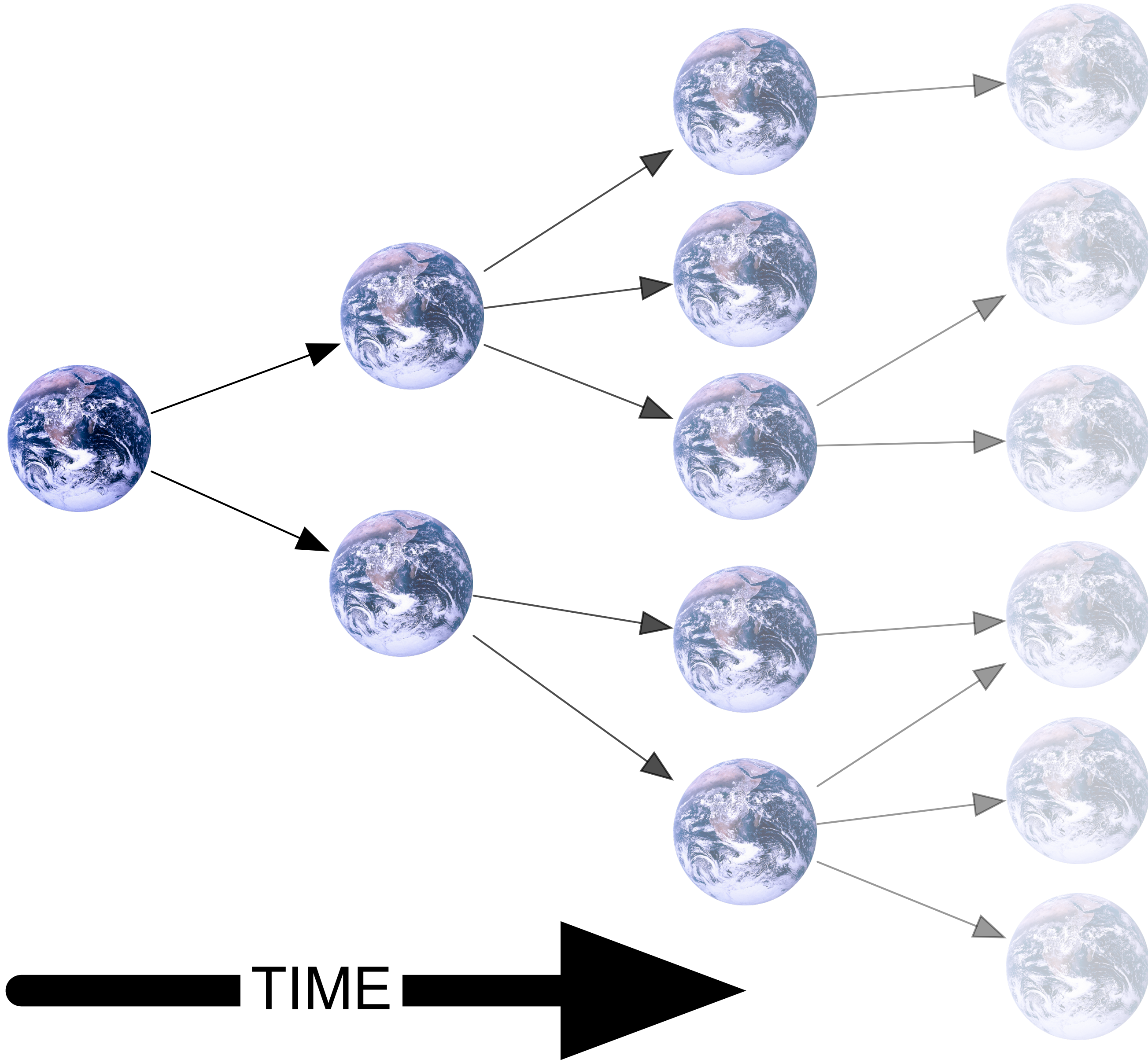
RZ Example: Exact Reals

work by Andrej Bauer and Istok Kavkler

- Axiomatize the (constructive) real numbers
- Run this through RZ
- Hand-write implementations that match the interfaces

Goal: An efficient implementation of exact real arithmetic with a strong theoretical basis.

Modal Logic: Logics of Possible Worlds

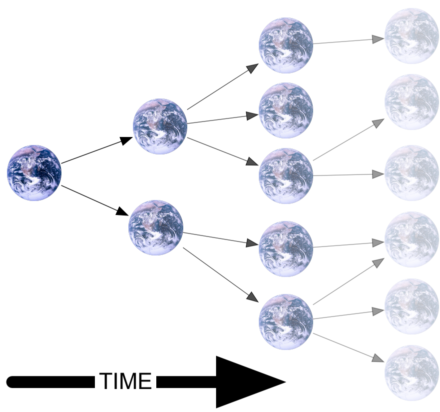


Propositions About Aliens

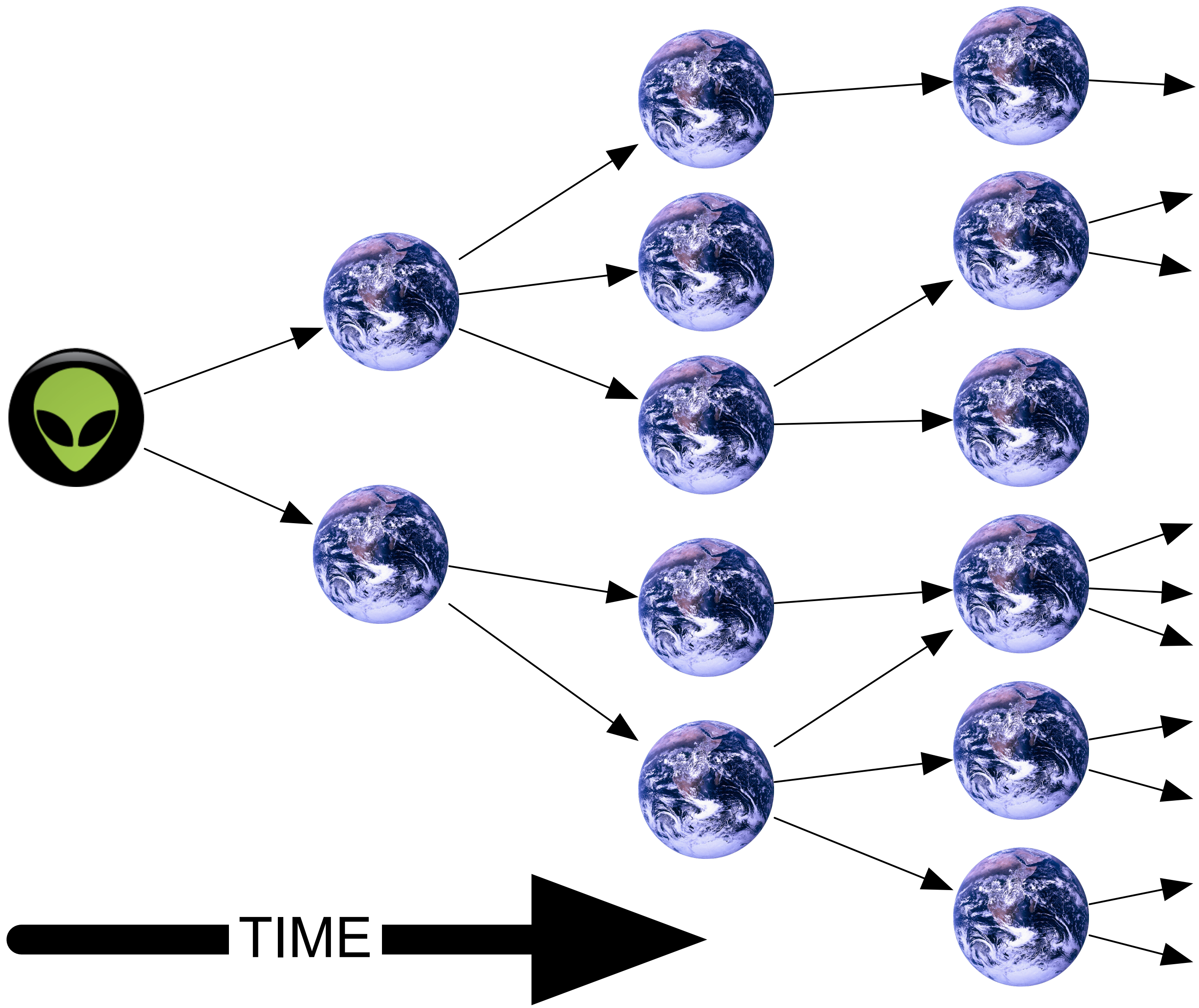
- Aliens are among us (now).
- Aliens may eventually be among us.
- Aliens might always be among us.
- Aliens could eventually be always among us.
- Aliens will always be among us.
- If faster-than-light travel is invented, aliens will eventually be among us.

New Quantifiers

	Eventually True	Invariably True
Some Path	EF p Possibly	EG p Potentially Always
All Paths	AF p Inevitably	AG p Invariably



Aliens



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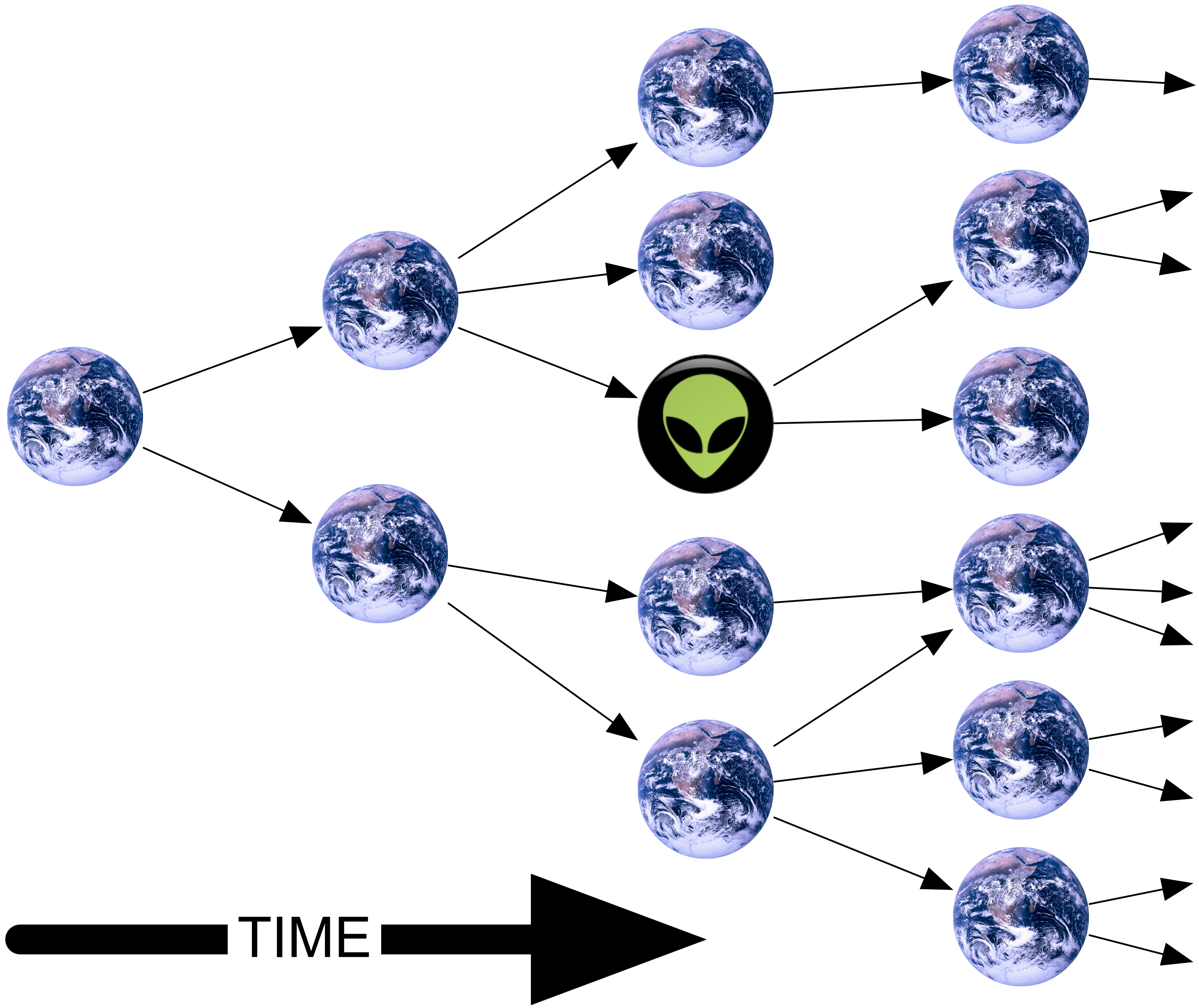

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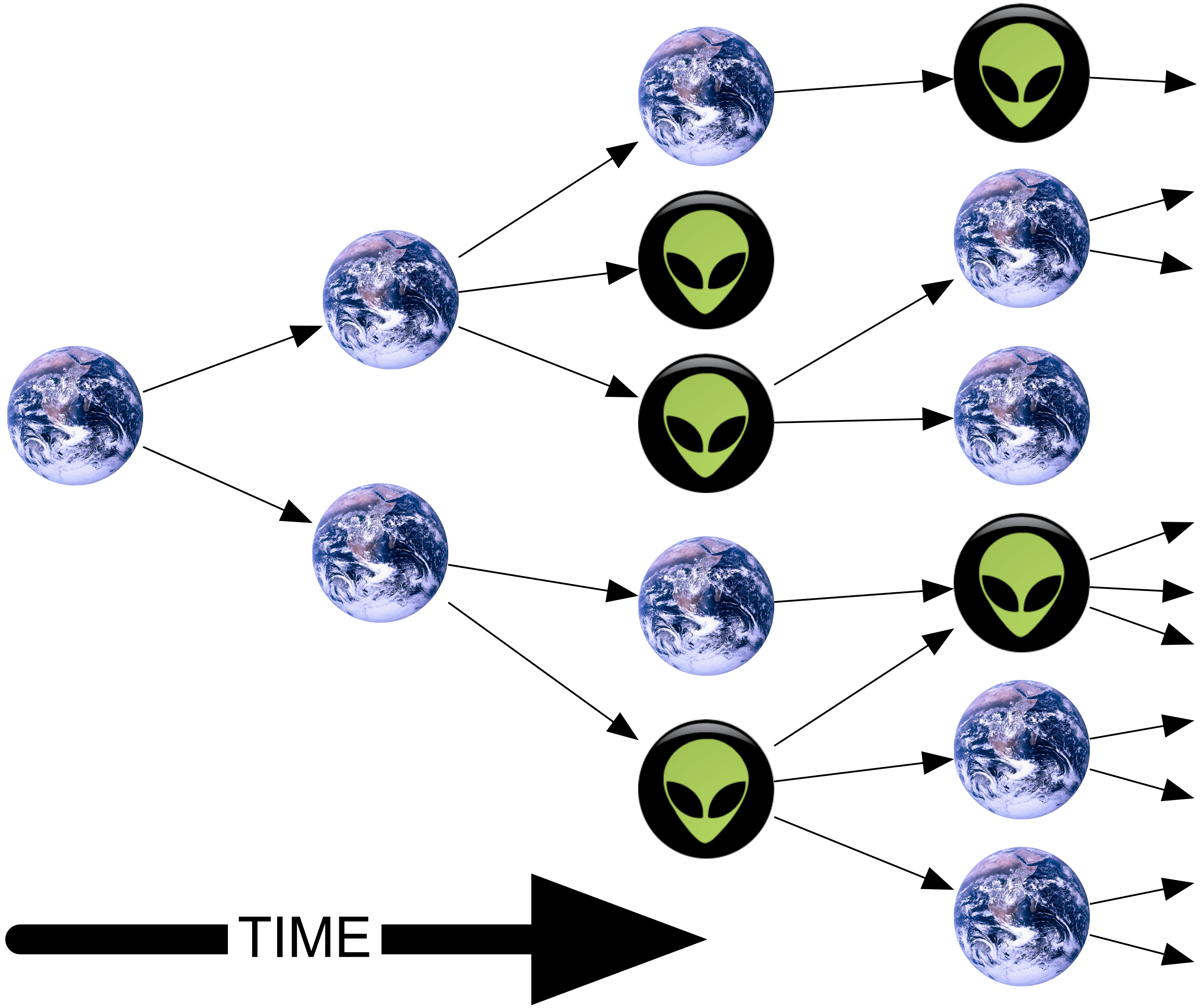

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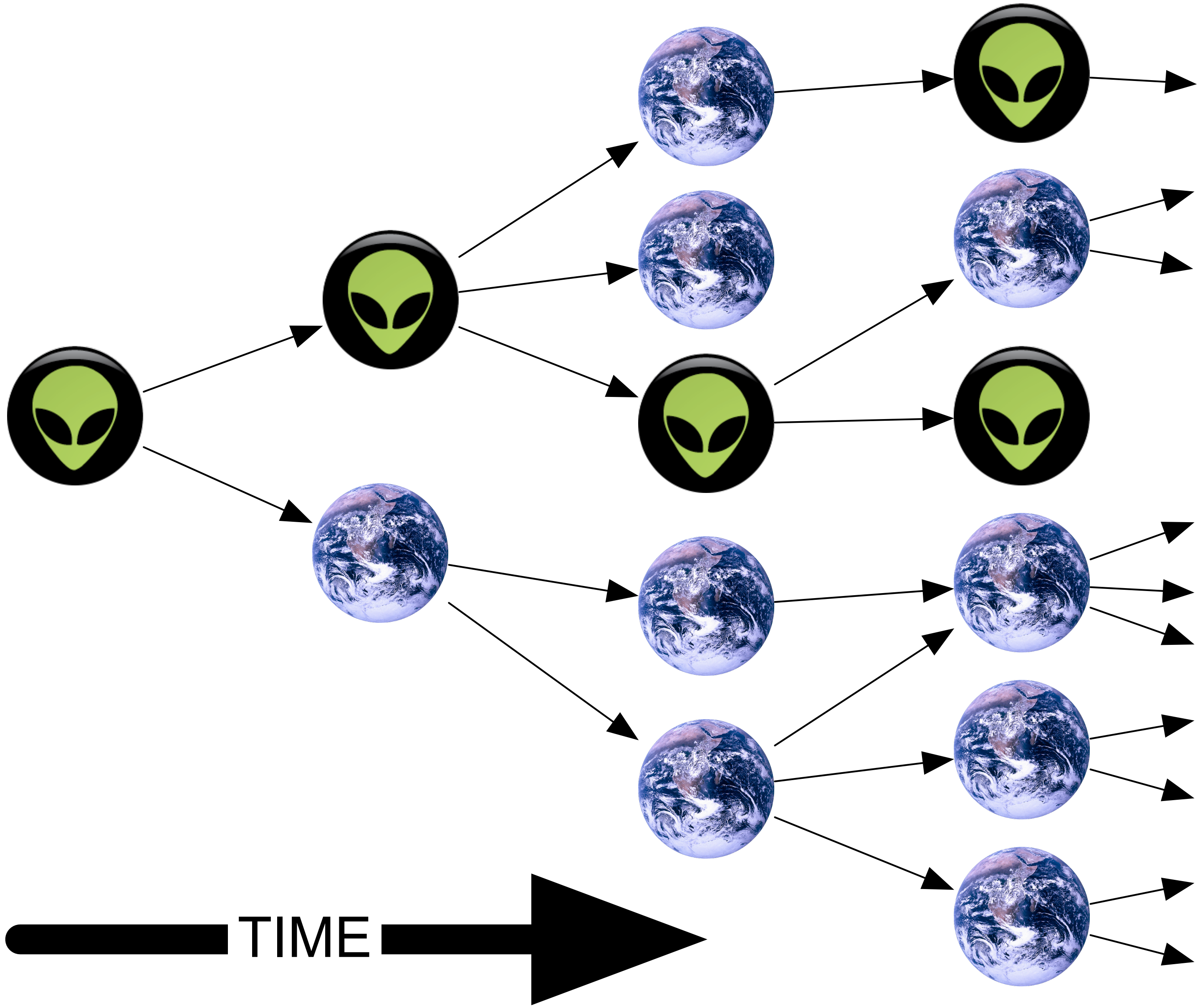

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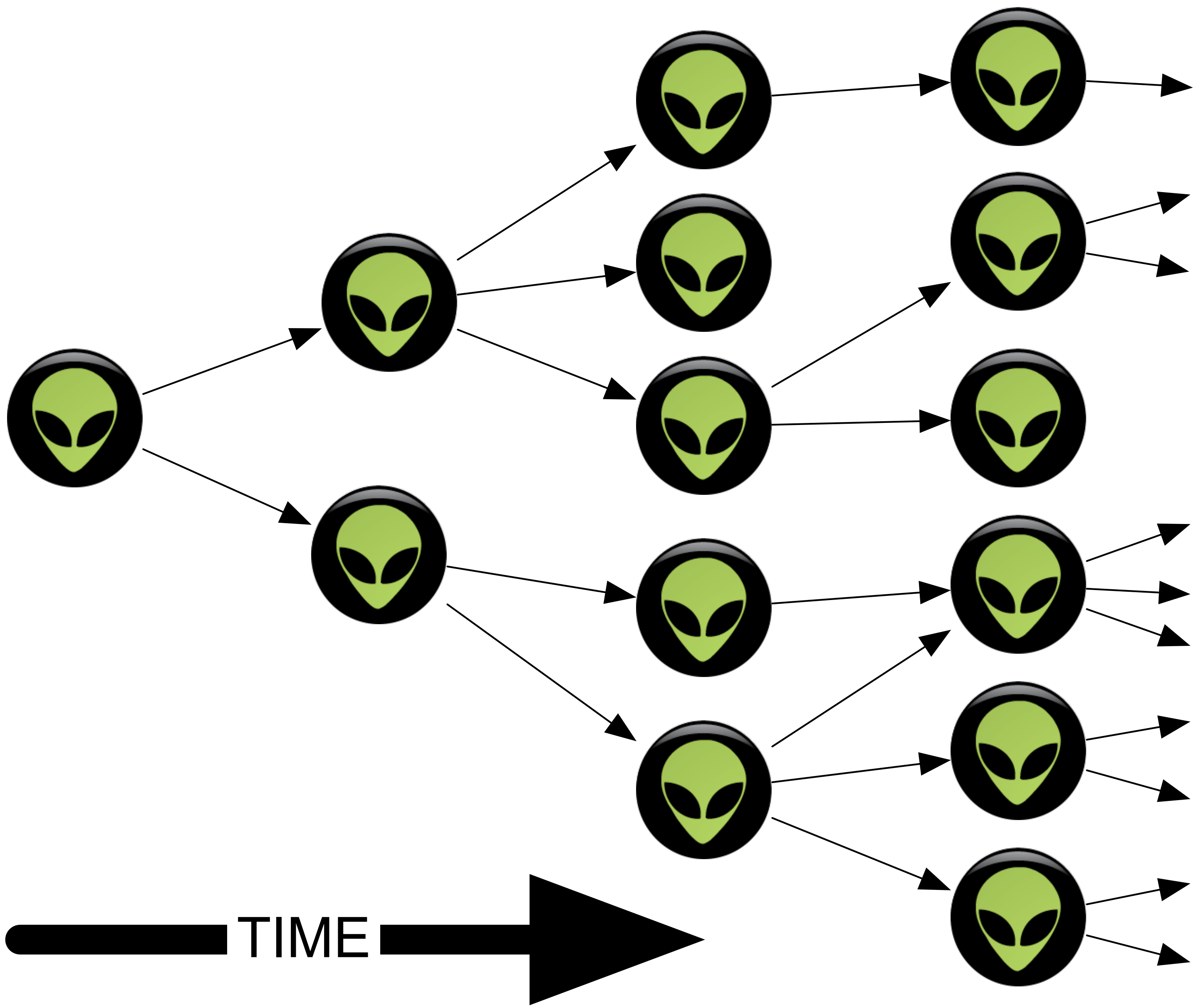
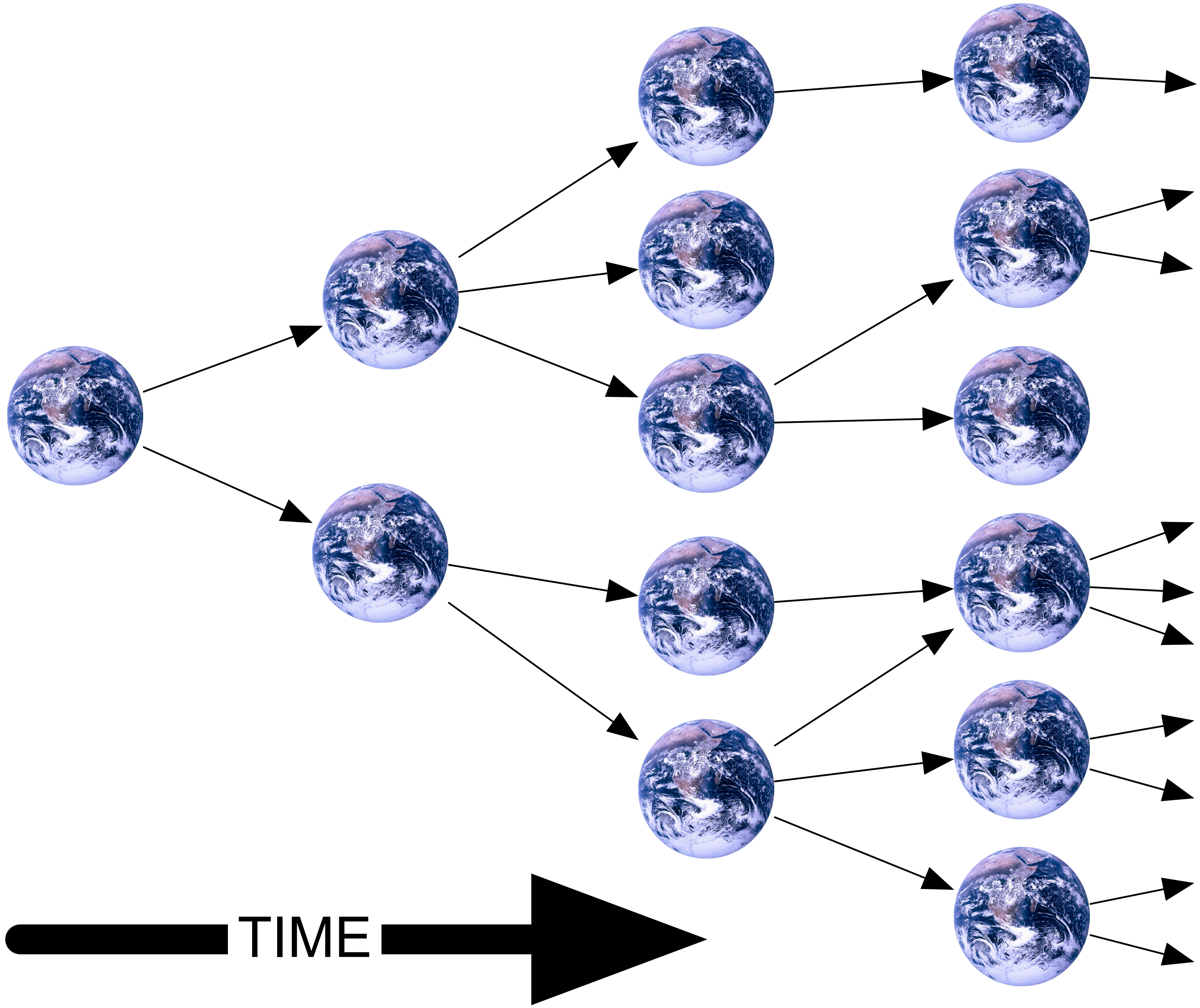
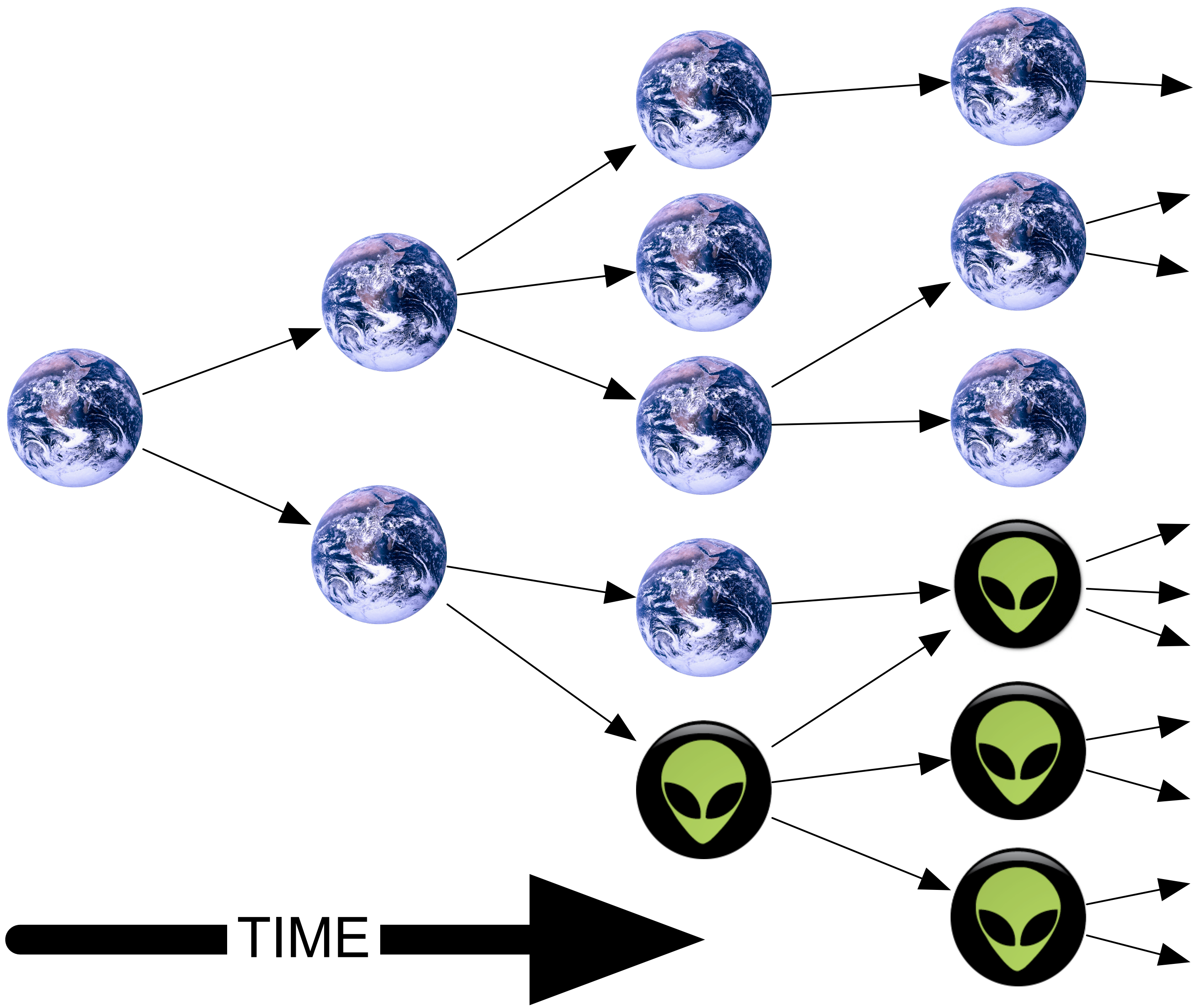


AG(not $Aliens$)



$\mathsf{EF}(\mathsf{AG}\ Aliens)$



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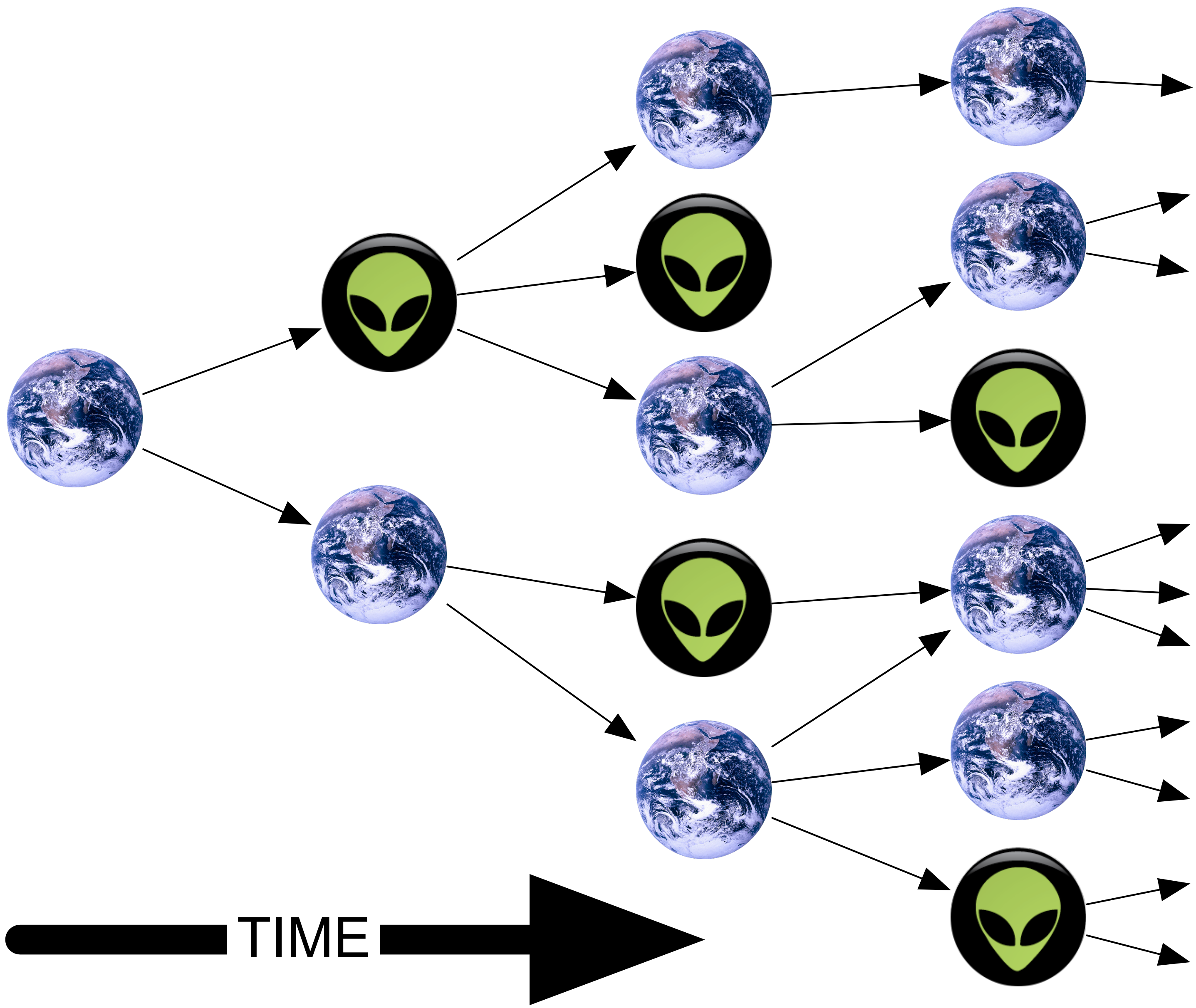

Model Checkers (Spin, SMV, Uppaal,. . .)

Particularly successful in hardware verification and protocol verification.

- Start with finite-state systems
  - Explicit states ($n = 3$) vs. symbolic states ($3 < x \leq 4$)
  - "Finite" includes millions or billions of states, or more
  - Transitions can be bits of computer code

- Automatically verify properties, using
  - Exhaustive search of reachable states (as necessary)
  - Clever representations (e.g., BDDs)
  - Abstractions

- Need a language of properties: usually some form of temporal logic

CSL: Checkable Sequence Language

with Bob Keller, Heather Justice (HMC '09), Daniel Furlong (HMC '11), Andrew Carter (HMC '12), Yu-Wen Tung (JPL)

- Spacecraft have constraints:
  - "Don't run both heaters at the same time"
  - "Never open the hatch during the launch phase"
  - "The antenna must be recalibrated at least once per hour"
  - "Don't power the thruster for more than one hour"

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  - Ignores nondeterminism/unknowns/failure modes
- CSL: a (Java-like) domain-specific language designed
  1. for spacecraft simulations
  2. to be model-checkable

Linear Logic: A Logic of Resources

Compare

If $\$10 \rightarrow$ book and $\$10 \rightarrow$ movie ticket, then
$\$10 + \$10 \rightarrow$ (book and movie ticket).

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If $p \rightarrow q$ and $p \rightarrow r$, then
$p \rightarrow (q \text{ and } r)$

Compare

$$\text{If } \$10 \to \text{book and } \$10 \to \text{movie ticket, then}$$
$$\$10 + \$10 \to (\text{book and movie ticket}).$$

$$\text{If } p \to q \text{ and } p \to r, \text{ then}$$
$$p \to (q \text{ and } r)$$

$$\text{If } \$10 \to \text{book and } \$10 \to \text{movie ticket, then}$$
$$\$10 \to (\text{book and movie ticket}).$$

Compare

If $\$10 \rightarrow$ book and $\$10 \rightarrow$ movie ticket, then
$\$10 + \$10 \rightarrow$ (book and movie ticket).

If $p \rightarrow q$ and $p \rightarrow r$, then
$p \rightarrow (q \text{ and } r)$

If $\$10 \rightarrow$ book and $\$10 \rightarrow$ movie ticket, then
$\$10 \rightarrow$ (book and movie ticket).

[If first prize in a raffle is a book; second prize a ticket]
winning-ticket $\rightarrow$ (book or movie ticket).

Linear Connectives

$p \otimes q$	Both p and q simultanously
$p \,\&\, q$	One of p or q (your choice)
$p \oplus q$	One of p or q (not your choice)
$p \multimap q$	q follows if I use p exactly once
$!p$	Zero or more copies of p, as needed

An Example (due to Patrick Lincoln)

Fixed-Price Menu: $5

Hamburger

Coke

Fries (All-you-can-eat)

Onion Soup or Salad

Dessert of the Day (Pie or Ice Cream)

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$$D \otimes D \otimes D \otimes D \otimes D$$

$$\multimap$$

$$H \otimes C \otimes !F \otimes (O \,\&\, S) \otimes (P \oplus I)$$

An Example (due to Patrick Lincoln)

Fixed-Price Menu: $5
Hamburger
Coke
Fries (All-you-can-eat)
Onion Soup or Salad
Dessert of the Day (Pie or Ice Cream)

$$D \otimes D \otimes D \otimes D \otimes D$$
$$\multimap$$
$$H \otimes C \otimes !F \otimes (O \mathbin{\&} S) \otimes (P \oplus I)$$

Note: The US Government gets to assume $!D$. You don't.

Sample Applications

Linear type systems for programming languages: make "proper" use of resources or your program won't compile/run:

- Memory used must be relinquished
- Disk files opened must be closed
- Resources cannot be ignored: use or release.

Describing Stateful Computation: A piece of program code like

```
x = x + 1
```

can be modeled as a function

$$\text{Memory} \to \text{Memory}$$

or, more accurately,

$$\text{Memory} \multimap \text{Memory}$$

Other applications: concurrency, linguistics, ...