Computation Histories and PCP

(not Probabilistically Checkable Proofs)
(also not Angel Dust)

CS 81: Computability and Logic

November 23, 2010

Computation History

- Step-by-step recording of a TM computation.
- Used to show more problems not decidable.

States as Strings

- We can describe the configuration of any TM using a string $C = xqy \in \Gamma^*$.
  - $x \in \Gamma^*$ = symbols to the left of head
  - $q \in Q$ = current control state
  - $y \in \Gamma^*$ = symbols under and to the right of head
- Over the course of a computation, we have
  - $\cdots \Rightarrow x_1q_1y_1 \Rightarrow x_2q_2y_2 \Rightarrow x_3q_3y_3 \Rightarrow \cdots$
  - If the TM halts, we can represent the whole history of the computation as a single (finite) string!
  - Traditionally written $\#C_1\#C_2\#C_3\#\cdots\#C_n\#$

Checking a History

- Checker: a Turing Machine C that, given <M, h>, checks whether h is a history of TM M.
  - Consecutive states should be equal, except around the head (where the change corresponds to the transition table of M).
- Can check whether h is a halting (or an accepting) history by looking at the last control state.

Digression: LBAs

- In fact, the CH can be checked for validity by a less-than-general TM called an LBA.
  - LBA = "Linear Bounded Automaton," a TM that can only use the part of the tape containing input.
  - An LBA can have a large tape alphabet and can "mark" tape cells. It just can't grow its tape.
- More powerful than a DFA
  - Number of potential states grows with input size
  - DFA wouldn't be able to check a computation history.

Accepting for LBAs

- A_{LBA} = { <M,w> | M a LBA accepting w } is decidable.
- Proof: When running M on w, there are at most

$$n := |Q| \times |\Gamma|^{|w|} \times |w|$$

  distinct "states" during the computation.
- So,
  - Run M on w.
  - If computation takes longer than n steps, its in an infinite loop; M doesn't accept w.

Emptiness for LBAs

- E_{LBA} = { <M> | M a LBA, L(M) = $\varnothing$ } isn't decidable.
- Proof: $A_{TM} \leq E_{LBA}$.
  -

All_{CFG}

All_{CFG} = { <G> | G a CFG, L(G) = Σ^* } is undecidable.

Proof: $A_{TM} \leq A_{CFG}$.

- Key: given <M,w> create a PDA/CFG for strings that **aren**'t accepting computation histories!
  - PDA accepts strings that
    - Don't start with q_0w
    - Or, don't end with $xq_{accept}y$
    - Or, two successive configurations don't match properly
  - <u>Hack</u>: need to reverse every other configuration.
- The grammar for this PDA is Σ^* iff M,w has no finite, accepting history.

Eq_{CFG}

$Eq_{CFG} = \{ \langle G_1, G_2 \rangle \mid L(G_1) = L(G_2) \}$ is undecidable.

Proof: $All_{CFG} \leq Eq_{CFG}$.

Post Correspondence Problem

Emil Post

- Named after logician Emil Post (1897-1954)
  - studied fundamental models of computation
  - "scooped" by Gödel, Turing, and Church

Why PCP?

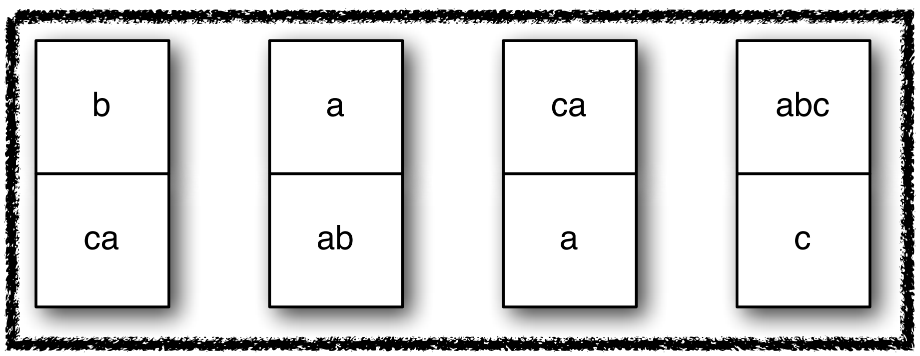
- Trivial problem to state
  - Looks nothing like Turing Machines
  - A child can understand it
  - Superficially, doesn't look that hard

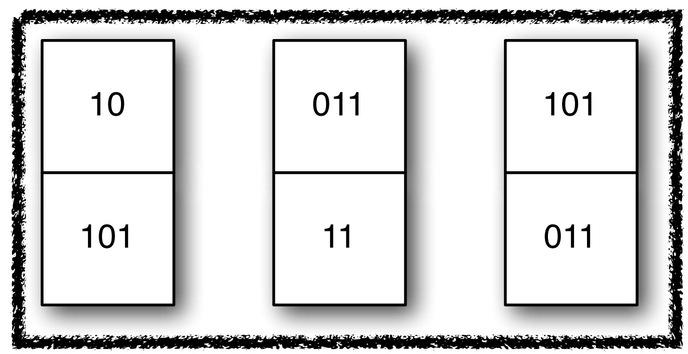
- Can reduce PCP to other problems, showing them undecidable

PCP

(not a Probabilistically Checkable Proofs)
(not Angel Dust)

- Given a set of "dominos" (pairs of strings), find a sequence of dominos so that the top strings and the bottom strings match.





- Theorem: PCP is undecidable.
- Proof: Halting $\leq$ MPCP $\leq$ PCP

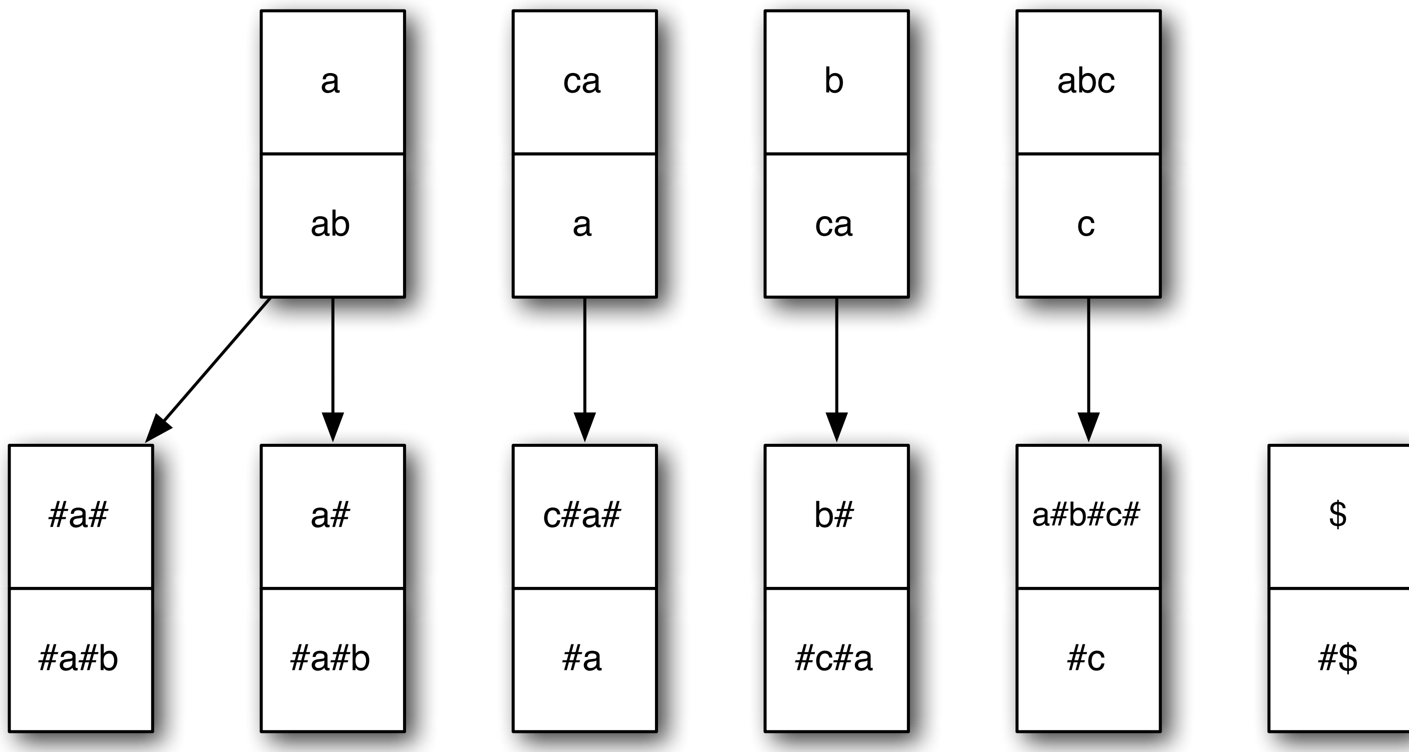
Modified PCP (MPCP)

- Like PCP, but solution starts with <u>first</u> domino.
- The following MPCP instance has no solution.

10	011	101	0
101	11	011	0

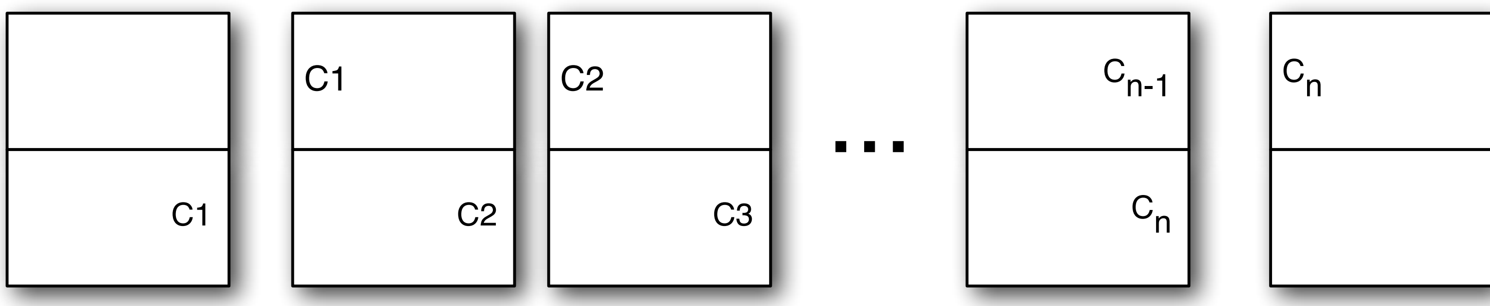
MPCP ≤ PCP

- Given an instance of MPCP, solve by translating dominos and doing PCP.



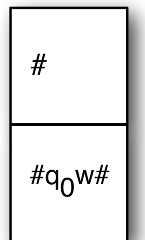
Halts ≤ MPCP

- Idea: if a TM halts, it has a computation history
- Given a <M,w>, construct dominos whose solution would yield such a history (on top and bottom); use MPCP-solver to check for a solution.

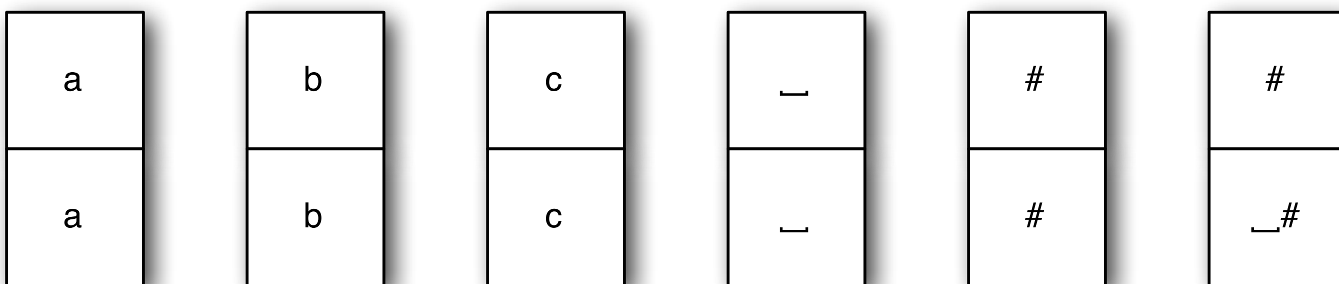


The Dominos

- First domino: set up the initial state

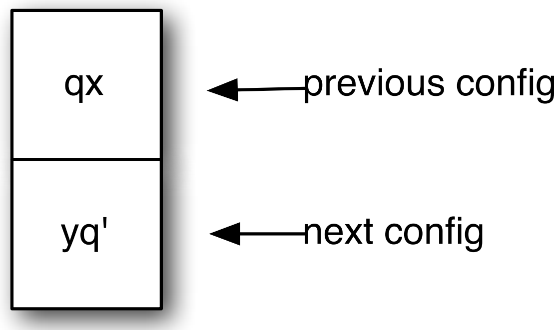


- Helper dominos: copy unchanged parts of the tape from C_i to C_{i+1}. (optional: expand the tape)

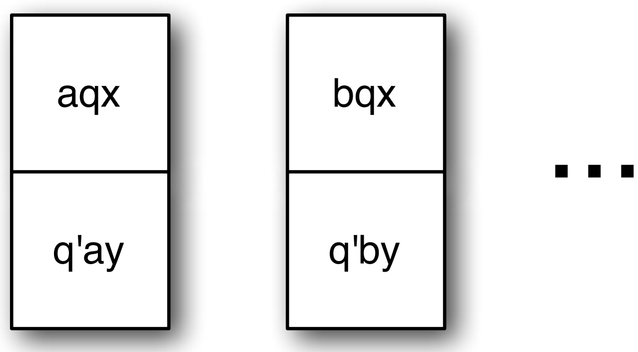


More Dominos

- For each transition $q,x \rightarrow q', y, R$

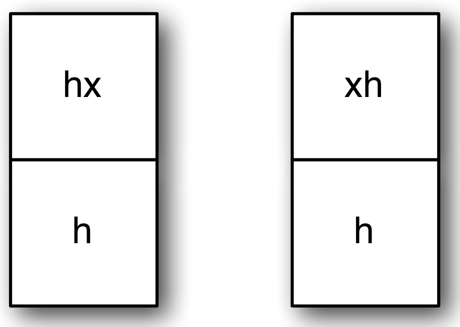


- For each transition $q,x \rightarrow q', y, L$

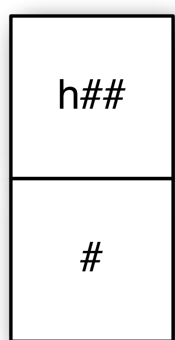


Completion Dominos

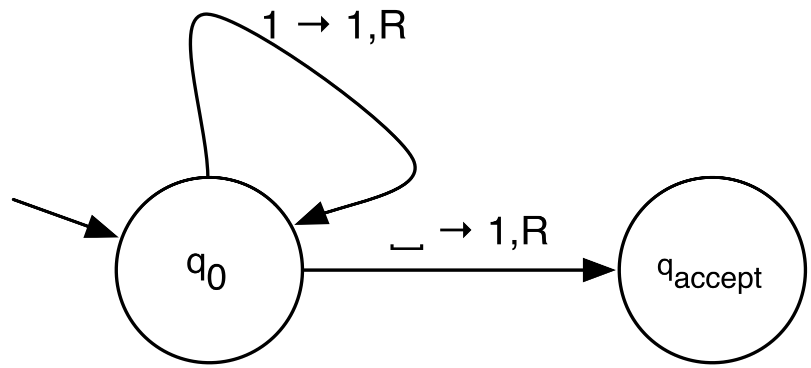
- Technical "Trick"
  - When we reach a halting state h, we delete the configuration (one symbol at a time) until it disappears
  - For each halting state h, and each symbol x, we need



  - Finally,



Example: w=11



#	1	␣	#	#
#$q_0$11#	1	␣	#	␣#

$q_0$1	q_0␣	1q_a	q_a1	q_a##
1q_0	1q_a	q_a	q_a	#