More TMs

CS 81: Computability and Logic

November 16, 2010

TM Languages

- A TM accepts a string if, given the string as input, the TM reaches q_{accept}.

- A language is recognizable (recursively enumerable) if there is a turing machine that accepts exactly the strings in the language.

- A language is decidable (recursive) if it is accepted by a TM that always halts (i.e., if the TM always ends up in q_{accept} or q_{reject}.)

TMs Computing Functions

- Rather than accepting a language, we can use a TM to **compute a function**:
  - The machine starts with some string x on its tape initially.
  - The machine halts with some string y on its tape finally.
  - The corresponding function f would have

    $$f(x) = y$$

  - If the TM doesn't halt, it computes a **partial** function.

Turing Machines as Enumerators

- Several variant definitions. Each specify a language L.
    1. A TM that prints out all the members of L, one at a time (but not necessarily in any particular order)
    2. A TM that prints out all the members of L, one at at time (but...) with arbitrarily many repeats.
    3. A TM that, given an integer n, returns the n^{th} element of a sequence like (1) above.
    4. A TM that, given an integer n, returns the n^{th} element of a sequence like (2) above.
- For a fixed language, all these are interconvertable.
- A language is recognizable iff it can be enumerated.

Encoding TMs as Strings

- We know that a TM can be described fully by a finite list of transition rules, of the form:
  - q_0, ␣ $\rightarrow$ q_i, ␣, R
  - q_i, ␣ $\rightarrow$ q_j, ␣, L
  - q_i, x $\rightarrow$ q_k, x, L
  - q_m, a $\rightarrow$ q_n, x, R etc.
- We can devise a way to encode an **arbitrary** set of such rules into a **fixed alphabet**.
- Although the number of states and tape symbols can be arbitrary
  - we can **encode** these by using strings of symbols, say {0, 1}
  - concatenate the symbols to describe rules
  - concatenate rules to describe machines.

Encoding TMs as Strings

- Not **every** string of symbols in the encoding alphabet must correspond to a well-formed TM description.
- But we can determine algorithmically which are and which aren't.
  - In fact, a TM can do so!
  - For those that aren't, assume they describe a default TM that immediately halts and rejects.
- We thus can enumerate **all TM's**:

  $$M_0, M_1, M_2, \ldots$$

- We can do the same for encodings of **initial tapes**:

  $$x_0, x_1, x_2, \ldots$$

- We can even enumerate all pairs: $\langle M_i, x_j \rangle$

Universal Turing

- An encoding of a Turing machine can be viewed as a *program*.
- A Turing machine that interprets such a program to carry out the actions specified is called a **Universal Turing Machine** (UTM).

Universal Turing

- UTMs can be shown to exist by constructing them.
- Think about what would be required.
  - The tape has to hold the tape of the machine being simulated.
  - The tape has to hold the program of the machine being simulated.
  - The program must be laid out in such a way that the necessary markers can be inserted to keep track of the current state, etc.
- All this is possible, if somewhat laborious to construct.

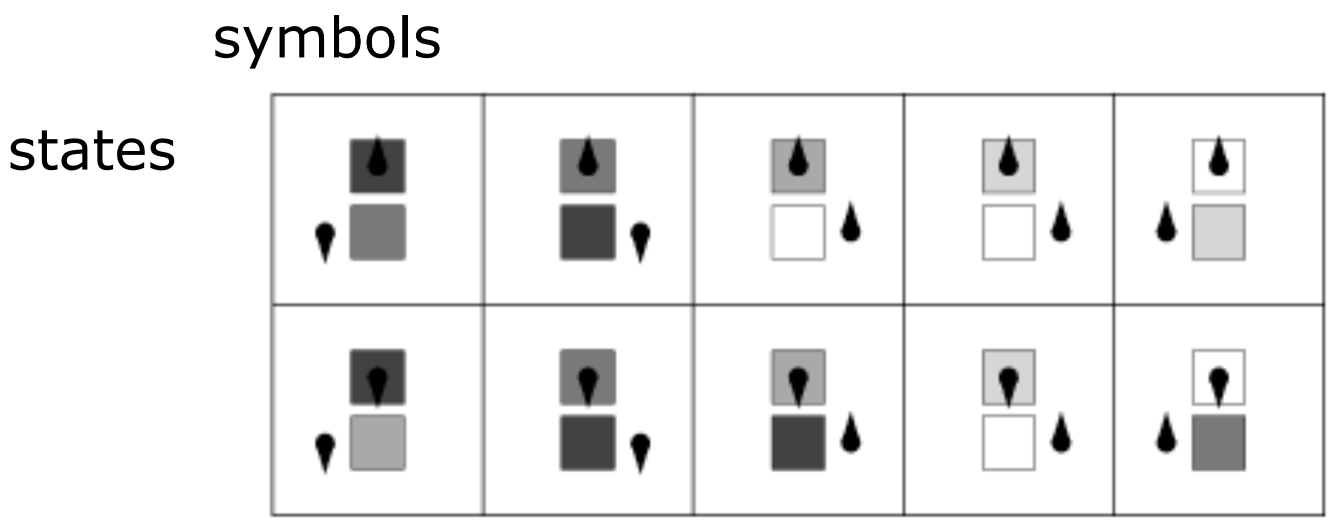
- Whether a machine is universal will depend on the particular encoding used.

Specific UTMs

- The first was constructed by Turing himself.
- Shannon showed any UTM could be converted either to a 2-symbol machine or to a 2-state machine by augmenting states or symbols, respectively.
- Minsky (1960) gave a 7-state 6-symbol machine.
- Watanabe (1961) gave an 8-state 5-symbol machine.
- Minsky (1962) gave a 7-state 4-symbol machine.
- Rogozhin (1996) gave a 4-state 6-symbol machine
- Wolfram and Reed (2002) gave a 2-state 5-symbol machine.
- Smith and Wolfram (2007) gave a 2-state 3-symbol machine.
- No 2-state 2-symbol UTM exists.

A Specific Universal Turing Machine

- 2-state, 5 symbol UTM published by **Wolfram** in 2002



adapted from Wolfram, S. *A New Kind of Science*. Wolfram Media, p. 707, 2002.

Computability and Uncomputability

Question

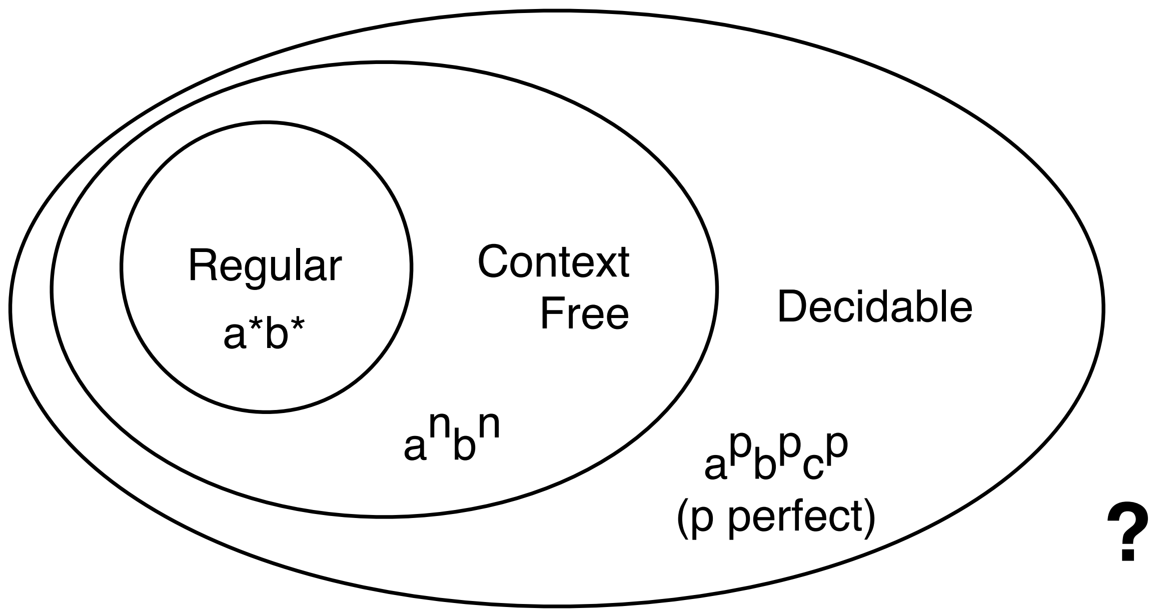
- How is my laptop more like a Finite State Machine than like a Turing Machine?

- How is my laptop more like a Turing Machine than like a Finite State Machine?

Church-Turing Thesis

- If it can be done at all, then it can be done by
  - A Turing Machine
  - Lambda Calculus
  - An Unrestricted Grammar
  - A 2-register machine
  - C
  - ...
- **Note**: assumes suitably coded inputs and outputs

Is There More?



Some Languages Aren't Decidable

- Given a finite Σ, how many languages are there?
- How many TMs are there?
- QED

But wait…

- How many languages over Σ could I describe (say, in a LaTeX document)?
- How many TMs are there?
- QED?

Languages of Acceptance

- Which are recognizable (by a TM)? Decidable?
  - $A_{DFA} = \{ \langle D,w \rangle \mid D \text{ a DFA, } D \text{ accepts } w \}$
  - $A_{NFA} = \{ \langle N,w \rangle \mid N \text{ an NFA, } N \text{ accepts } w \}$
  - $A_{RE} = \{ \langle R,w \rangle \mid R \text{ a regexp, } R \text{ matches } w \}$
  - $A_{CFG} = \{ <G,w> \mid G \text{ a CFG, } G \text{ produces } w \}$
  - $A_{TM} = \{ <M,w> \mid M \text{ a TM, } M \text{ accepts } w \}$

Digression: Bootstrapping a Compiler

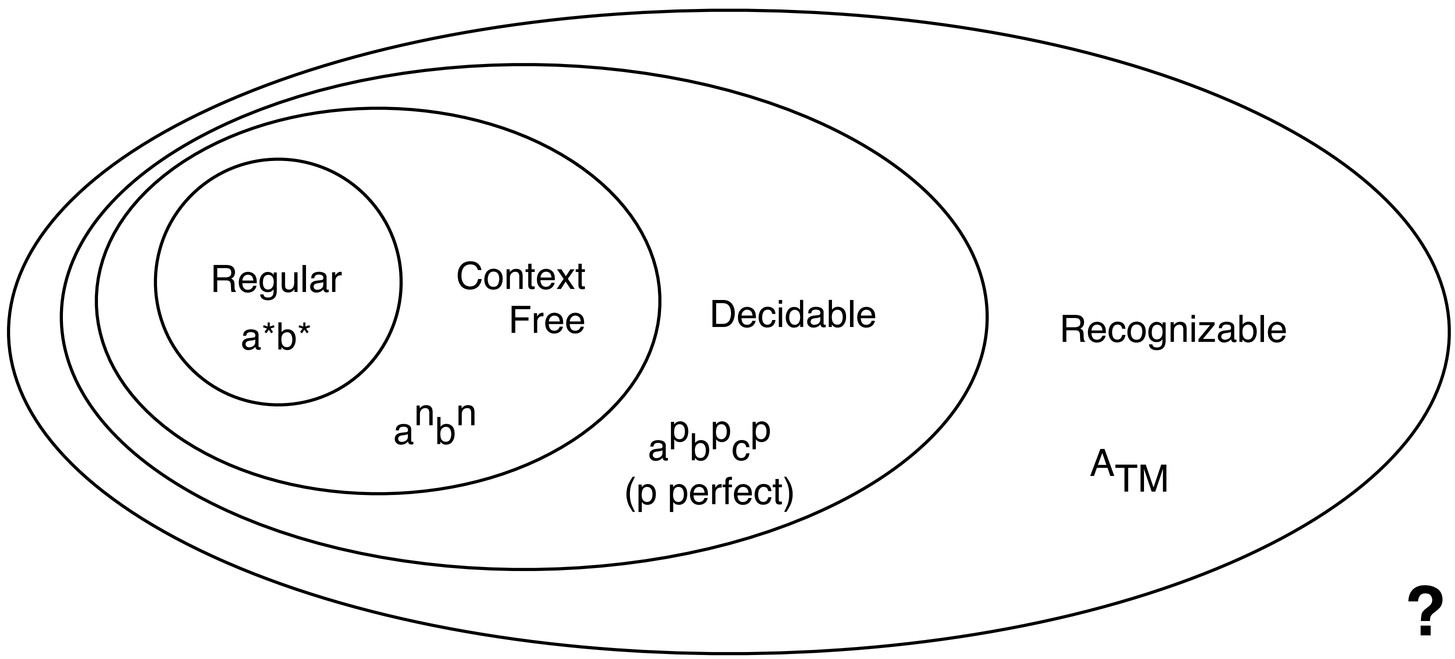
- Lots of compilers are written in the same language they compile!
  - Gnu C Compiler (used in CS 105) is written in C
  - Glasgow Haskell Compiler (used in CS 131) is in Haskell
  - etc.

- Consequence:
  - We sometimes run programs (compilers) on their own source code!

A_{TM} is not decidable

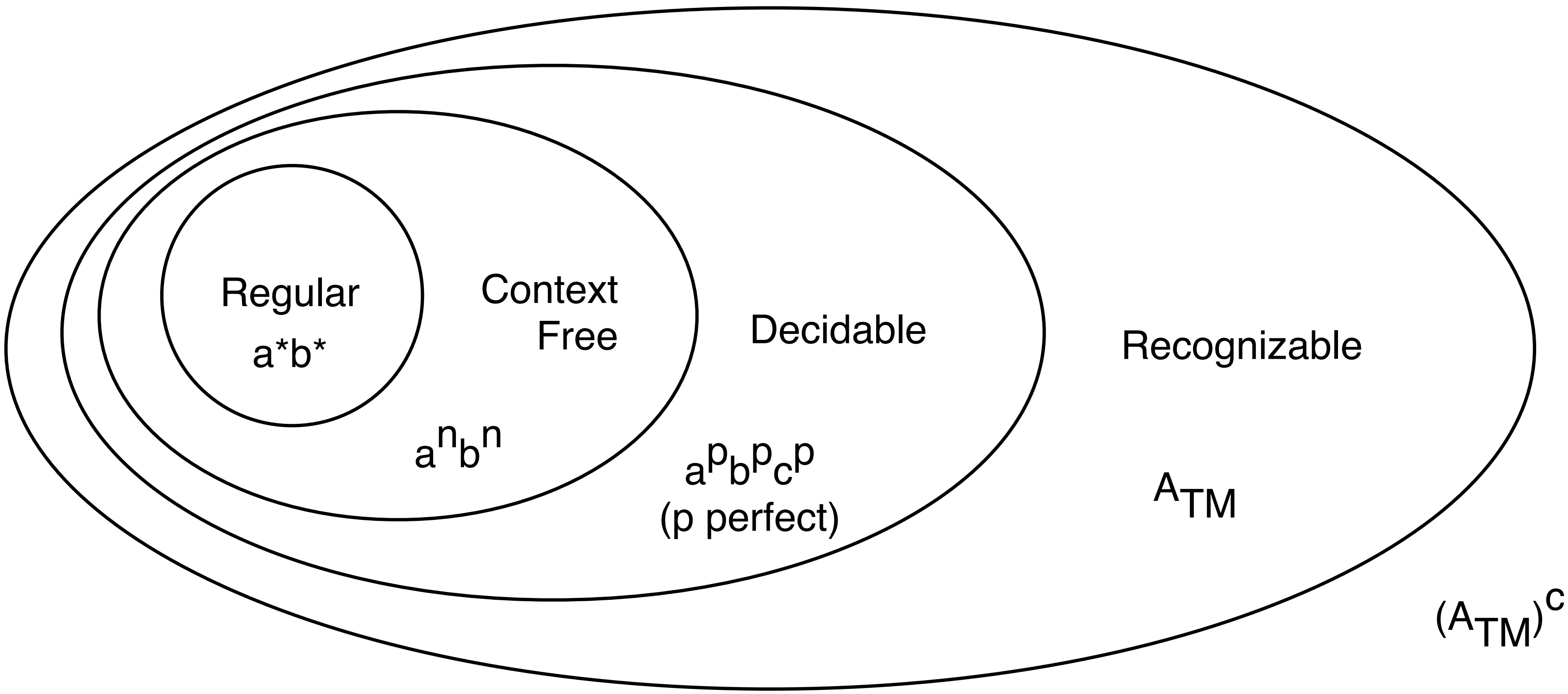
	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$
M_0	Acc.		Acc.		Acc.	
M_1						
M_2	Acc.			Acc.		
M_3	Acc.	Acc.	Acc.	Acc.	Acc.	
M_4			Acc.	Acc.	Acc.	
$\vdots$						

Is There More?



Recall:

- If L and L^c are both recognizable, then they are both decidable.
- What is the complement of A_{TM}?


Regular
a*b*
Context Free
$a^n b^n$
Decidable
$a^p b^p c^p$
(p perfect)
Recognizable
A_{TM}
$(A_{TM})^c$

Obligatory Corollary

- The language

  $H = \{ \langle M,w \rangle \mid M \text{ a TM that halts given } w \}$

  is not decidable.

- Proof: Suppose there <u>were</u> a halt-checking TM....