All proofs are informal and intended to develop intuition and expository style, so please provide convincing write-ups of each.

1. [5 points] What is the smallest alphabet $\Sigma$ such that all Turing machines can be encoded as members of $\Sigma^*$?

2. [20 points] Consider the language of Turing machine descriptions $L = \{<M> | M \text{ accepts at least 999 strings}\}$. Is $L$ decidable, recognizable, or neither? Prove your answer.

3. [10 points] Suppose $R \subseteq \Sigma^*$ is a regular language other than $\emptyset$ and $\Sigma^*$. Let $L \subseteq \Sigma^*$ be any decidable language. Show that $L \leq_m R$ (L is mapping-reducible to R).

4. [5 points] Regarding the previous problem, is the same reduction true if $L$ is recognizable but not decidable? Why or why not?

5. [20 points] Consider the language of Turing machine descriptions $L = \{<M> | M \text{ accepts all strings of length 999 or longer}\}$. Is $L$ decidable, recognizable, or neither? Prove your answer.

6. [15 points] Show that the special case of PCP (Post’s Correspondence Problem) over a 1-letter alphabet is decidable.

7. [10 points] Show that a language $L$ is recognizable iff $L \leq_m A_{TM}$ (the acceptance language for Turing machines).

8. [15 points] Consider the language of Turing machine descriptions $L = \{<M, w> | M \text{ accepts } w \text{ using at most 999 tape cells}\}$. Show that $L$ is decidable. (Hint: For how many steps can $M$ run without either accepting, rejecting, using more than 999 cells or going into an infinite loop?)