Computation Histories

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What is this?

- A computation history (CH) is a recording of the history of a Turing-machine computation.

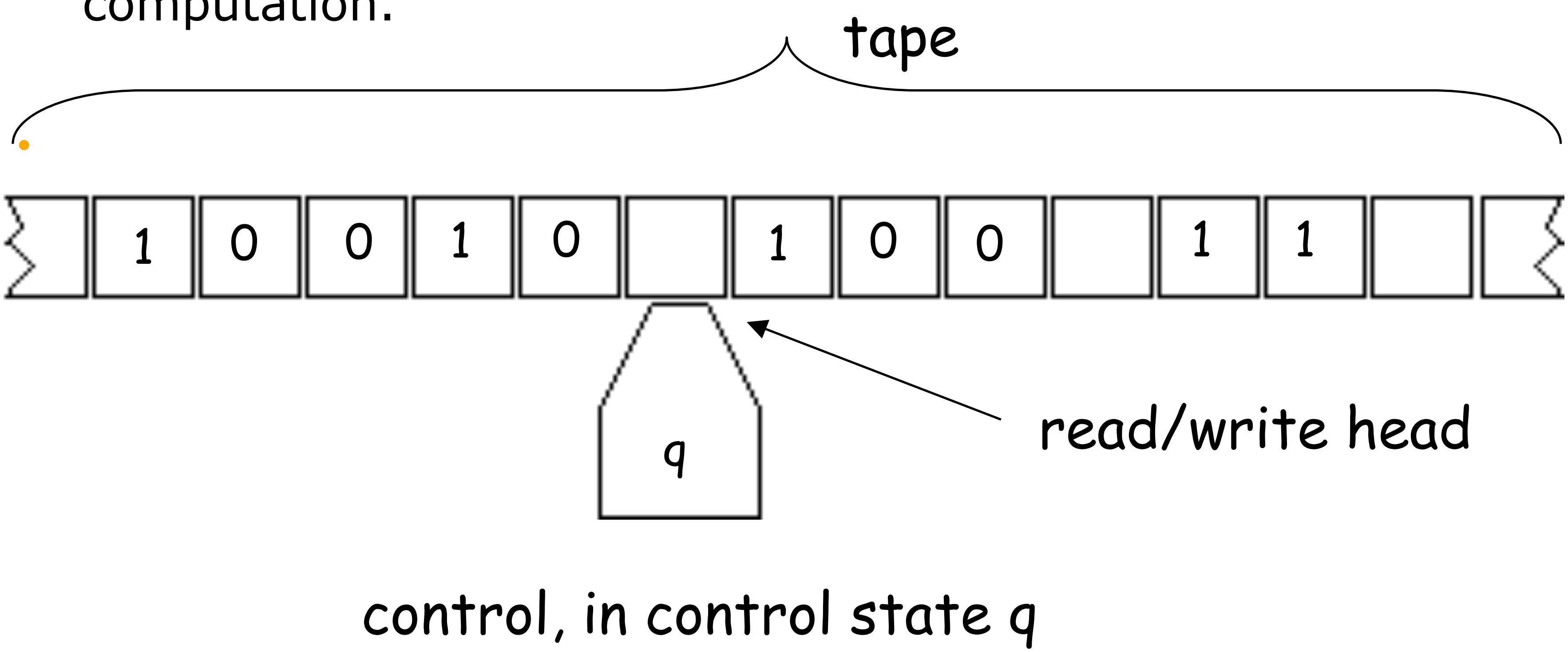
Of what use is it?

- Using CH’s we can demonstrate that various problems are undecidable.

- These problems would be difficult to show using, say, mapping reduction.

State of a Turing Machine

- The total **state** (often called a “configuration” to distinguish it from the control state) represents everything that needs to be known about the machine to continue the computation.



Capturing States as Strings

- We have seen this before:
- A state can be represented as a string
  $xqy \in \Gamma^*$
  where
  - Γ is the tape alphabet
  - x is the tape to the left of the head
  - q is the control state
  - y is the tape under and to the right of the head

Computation History

- As the TM computes, the states make transitions:

  $$x_1q_1y_1 \Rightarrow x_2q_2y_2 \Rightarrow x_3q_3y_3 \Rightarrow \ldots$$

- A deterministic machine will eventually either:
  - **Halt**: Reach a state xq_hy where no further transition is defined, or
  - **Diverge**: Never reach such a state, which means it goes on forever.

Encoding An Entire History

- **In the halting case**, the entire history is encodable as a **single string**:

  $x_1q_1y_1 \Rightarrow x_2q_2y_2 \Rightarrow x_3q_3y_3 \Rightarrow \ldots\ x_hq_hy_h$

  where we consider $\Rightarrow$ to be a symbol in a larger alphabet.

- Usually # is used in place of $\Rightarrow$, maybe because its easier to type.

Checking a History with a TM

- There is a Turing machine C (checker) that, with input <M, h>, where <M> is an encoding of an arbitrary TM, can check whether h is a history of M.

- All C needs to do is see if the symbols around the head of each state change correspond to the transition table of M.

- Further more, C can check whether h is a halting (or an accepting) history by looking at the last control state.

Checking a History with a TM

- $x_1a_1q_1b_1y_1 \Rightarrow x_2a_2q_2b_2y_2 \Rightarrow x_3a_3q_3b_3y_3 \Rightarrow \ldots$
  where each a_i, $b_i \in \Gamma$

  Do the local transitions $a_iq_ib_i \rightarrow a_{i+1}q_{i+1}b_{i+1}$ correspond to the rules in the description?

  Do the other parts of the string match?

  A TM can check these things.

LBAs

- In fact, the CH can be checked for validity by a less-than-general TM called an LBA.
- LBA = “Linear Bounded Automaton”,
  a TM that can never exceed the amount of tape on which the input is written.
- An LBA can have a tape alphabet that is bigger than the input alphabet, so it can effectively “mark” tape cells, etc.
  It just can’t **grow** its tape.

LBA vs. DFA

- At first glance, it might appear that an LBA is no more powerful than a DFA, since it cannot add new states.
- The difference is, however, that an LBA's total state set is a (linear) **function of the input string size**, which is not true for a DFA.

An Aside: 2-way DFAs

- If the LBA never writes on its tape, then it becomes a 2-way DFA.

- In this case, the power is reduced to that of a DFA: 2-way DFA’s only accept regular languages.

- The proof of this is non-trivial, and surprising. It involves the Myhill-Nerode theorem.

LBA for CH checking

- It is easy to see that an LBA can check a CH for correctness.

- It is also easy to see that no DFA could do this. It would require “matching” arbitrarily-long sub-strings. The Myhill-Nerode theorem or the pumping lemma could be used to prove this.

A_{LBA} is the Acceptance Language for LBAs

- Define A_{LBA} =

  $\{\langle M, w\rangle \mid M \text{ is an LBA} \wedge M \text{ accepts } w\}$

Theorem: A_{LBA} is decidable.

- Proof: Given an LBA encoding with tape <M, w>, we can simulate M on w.
- The maximum number of distinct tape states is $n^{|w|}$, where n is tape alphabet size and $|w|$ means the length of w.
- The number of different head positions is $1+ |w|$.
- The number of control states is m, say.
- So the total number of different states of M for an input w is $mn^{|w|}(1+ |w|)$.

(continued)

Proof that A_{LBA} is decidable (cont'd)

- For each step in the simulation of M on w, we increment the count, having started at 0.
- If M on w is still computing after $mn^{|w|}(1+ |w|)$ steps, we know that it is in a loop. So the simulating machine can **reject** <M, w> if this happens.
- The simulating machine will otherwise accept or reject <M, w>, depending on what happens with M on w.
- Hence there is a TM that can decide A_{LBA}.

Theorem: E_{LBA} is *un*decidable.

- Define the Emptiness Language E_{LBA} =

  {<M> | M is an LBA $\wedge$ L(M) = $\emptyset$}

  Proof: We show $A_{TM} \leq_m E^c_{LBA}$.

  Define f(<M, w>) = M′, where M′ is an LBA that accepts the **accepting computation histories** for M(w). Then L(M′) $\neq \emptyset$ iff <M, w> $\in A_{TM}$.

ALL_{CFG} is undecidable

- Define ALL_{CFG} =

  $\{<G> \mid G \text{ is a CFG} \wedge L(G) = \Sigma^*\}$

  We know that for every CFG, there is a corresponding PDA that accepts the same language.

  We show that PDA's can also, in a sense, check computation histories.

  The halting computation of a TM will be identified with the **absence** of a string accepted by the PDA.

Proof of ALL_{CFG} is undecidable

- For any TM and input <M, w> we construct a PDA that accepts all strings that are **not** valid computation histories.

- Thus, if the PDA accepts all strings, then M does not halt on w.

Proof of ALL_{CFG} is undecidable

- We represent the computation histories slightly differently in this case:
  Every other state is reversed.

- Originally: $x_1a_1q_1b_1y_1 \Rightarrow x_2a_2q_2b_2y_2 \Rightarrow x_3a_3q_3b_3y_3 \Rightarrow \ldots$

- Now $x_1a_1q_1b_1y_1$ # $\overline{x_2a_2q_2b_2y_2}$ # $x_3a_3q_3b_3y_3$ # …

- where overbar represents the reversal of the string below.

Why reverse?

- Reversing every other state enables two successive states to be checked by a PDA.
- The PDA, given an input, checks one of these (non-deterministically):
  - Does the input ***not*** begin with the initial state of <M, w>? If not, **accept**.
  - Does the input ***not*** end with an accepting state of <M, w>? If not, **accept**.
  - Starting at any one of the sections $C_i\#\overline{C_{i+1}}$ or $\overline{C_i}\#C_{i+1}$, do the two sections ***not*** represent consecutive states. If not, **accept**.
- So a string is **accepted** iff it does **not** represent a valid computation history.
- Thus the PDA **accepts** Σ^* iff M(w) does not have an accepting computation history.

Corollary: EQ_{CFG} is undecidable

- Define EQ_{CFG} =

  $\{\langle G, H \rangle \mid G \text{ and } H \text{ are CFG's} \wedge L(G) = L(H)\}$

  Proof: Show $ALL_{CFG} \leq_m EQ_{CFG}$.