Machines and Languages

Robert Keller
March 2010

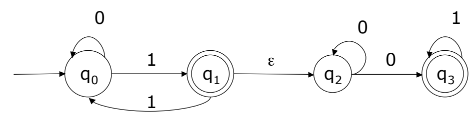
Machines

- By a **machine** over an alphabet Σ, we mean
  - a collection of **states** Q
    together with
  - a **transition relation** $-\rightarrow \subseteq Q \times (\Sigma \cup \{\varepsilon\}) \times Q$
    - $q - \sigma \rightarrow q'$ means that (q, σ, q') is in the relation
  - an **initial state** $q_0 \in Q$
  - and a set of **accepting states** $F \subseteq Q$.

Behavior of a Machine

- A machine **starts** in state q_0.
- From a current state q it can **change state** to q' **with input** σ provided that
  $q - \sigma \rightarrow q'$.
- From a current state q it can **change state** to q' **spontaneously** provided that
  $q - \varepsilon \rightarrow q'$.
- The machine **accepts** a string $x \in \Sigma^*$ provided there is *some* path from q_0 to *some* $q \in F$ that **spells out** x.

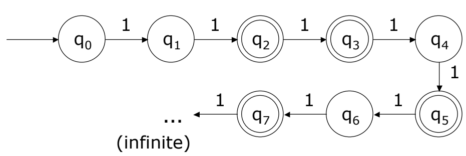
Example of a Machine



Accepted: 0 0 1 0 0 1

Not accepted: 1 0 1 0

Another Example of a Machine



Accepted: 1^p where p is prime

Not accepted: 1^q where q is composite

Languages for Machines

- If M is a machine, then L(M) is the **language accepted by** M, defined as the set of finite strings spelled out by all paths from the initial state to some accepting state.

The Language of a Machine *State*

- If q is a state, then the language L_q is defined to be the set of strings spelled out in going from q to some accepting state.
- Hence the language for the initial state is the language for the machine.

Equivalence of States

- Two states are defined to be **equivalent**

  $q \equiv q'$

  iff their languages are **equal**:

  $L_q = L_{q'}$.

Equivalence Relations Review

- $\equiv$ is an equivalence relation, meaning:
  - $\forall q \in Q \ (q \equiv q)$ [We drop the $\in Q$ for brevity below.]
  - $\forall q \ \forall q' \ (q \equiv q' \to q' \equiv q)$
  - $\forall q \ \forall q' \ \forall q'' \ ((q \equiv q' \wedge q' \equiv q'') \to q \equiv q'')$

Partitions

- Every equivalence relation determines a partition on Q.
- A partition is a set of subsets of Q such that:
  - No two subsets intersect.
  - The union of the subsets is all of Q.
- The partition determined by $\equiv$ is given by $\{\{q' \mid q' \equiv q\} \mid q \in Q\}$.
- The elements of the partition, sets of the form $\{q' \mid q' \equiv q\}$ are called the **equivalence classes** of $\equiv$.

Partition Determines Relation

- Every partition determines an equivalence relation:

  If P is a partition, then define: $q' \equiv q$ iff $\exists S \in P$ $(q \in S$ and $q' \in S)$.
- Verify that the 3 equivalence relation properties hold.

Rank of a Partition

- The rank of a partition is just the number of equivalence classes.
- If the state set is finite, the rank of the corresponding equivalence partition is also finite.
- If the state set is infinite, the rank could still be finite.

Machines for Languages

- A language $L \subseteq \Sigma^*$ can be viewed as a machine:
  - The states are elements of Σ^*.
  - The initial state is ε.
  - The transitions are $x - \sigma \rightarrow x\sigma$.
  - The accepting states are the elements of L.
  - As with any machine, there are languages L_x for each x in Σ^*. Incidentally, $L_x = \{ z \mid xz \in L\}$

Equivalence of Strings modulo a Language

- With respect to a language L, two strings are equivalent
  $x \equiv_L y$
  iff the languages of their corresponding states are equal:

  $L_x = \{ z \mid xz \in L\} = \{ z \mid yz \in L\} = L_y$

Congruence

- The relation $\equiv_L$ has the additional property of being a **congruence**:

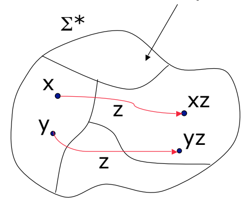
  $x \equiv_L y$ implies $\forall z \in \Sigma^* \ (xz \equiv_L yz)$.

- By induction, a necessary and sufficient condition for $\equiv_L$ to be a congruence is:

  $x \equiv_L y$ implies $\forall \sigma \in \Sigma \ (x\sigma \equiv_L y\sigma)$.

Congruence Pictured

Equivalence classes



Applying the same input sequence to congruent states yields congruent states.

Partition of the congruence

Finite-State Machines (FSMs)

- A machine is finite-state iff its state set is finite.

Determinism

- A machine is deterministic iff:
  - There are no spontaneous state changes, and
  - For each $q, q', q'' \in Q$ and $\sigma \in \Sigma$

    if $q - \sigma \rightarrow q'$ and $q - \sigma \rightarrow q''$, then $q' = q''$.

  In other words, there is a **partial function** $\delta: Q \times \Sigma \rightarrow Q$ such that $q - \sigma \rightarrow q'$ iff $\delta(q, \sigma) = q'$.

  "Partial" means that $\delta(q, \sigma)$ could be **undefined** for some pairs (q, σ).

DFA

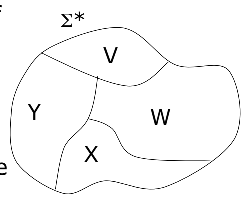
- A finite-state machine that is also deterministic is called a DFA (for deterministic finite-state acceptor).

Finite-State Languages

- Say a language L is finite-state iff it is accepted by some DFA.
- The equivalence partition for a finite-state language is guaranteed to be finite rank.
- Conversely, if the equivalence partition for a language is finite rank, then there is a DFA that accepts that language.

Language in terms of Equivalence Classes

- Consider a finite-state language.
- Its equivalence partition must be finite-rank.
- The language itself must be the union of some of the equivalence classes.



e.g. V∪Y could be the language

Myhill-Nerode Theorem

- $L \subseteq \Sigma^*$ is a finite-state language iff
- L is the union of some equivalence classes of some congruence relation on of finite rank.

Proof of Myhill-Nerode

- Suppose L is a finite-state language.
- Let M be a DFA accepting L.
- Let M be the a DFA.
- Let δ be the state-transition function described earlier, i.e. $\delta(q, \sigma) = q'$ means there is a transition from q to q' via letter σ.
- Extend $\delta: Q \times \Sigma \to Q$ to $\delta^*: Q \times \Sigma^* \to Q$, as follows:
  - $\forall q \in Q$ $\quad \delta^*(q, \varepsilon) = q$
  - $\forall q \in Q\ \forall x \in \Sigma^*\ \forall \sigma \in \Sigma$ $\quad \delta^*(q, \sigma x) = \delta^*(\delta(q, \sigma), x)$
- Claim: $x \equiv_L y$ iff $\delta^*(q_0, x) \equiv \delta^*(q_0, y)$.

Lemma

- $\forall z \in \Sigma^*\ \forall x \in \Sigma^*$
  $\forall q \in Q\ \delta^*(q, xz) = \delta^*(\delta^*(q, x), z)$
- Proof is by induction on x.
- Basis $x = \varepsilon$: $\delta^*(q, \varepsilon) = q$ and $xz = z$, so $\delta^*(q, xz) = \delta^*(q, z) = \delta^*(\delta^*(q, \varepsilon), z)$

Lemma

- Induction step:

  Assume $\forall q \in Q\ \delta^*(q, xz) = \delta^*(\delta^*(q, x), z)$.

  Show $\forall q \in Q\ \forall \sigma \in \Sigma\ \ \delta^*(q, (\sigma x)z) = \delta^*(\delta^*(q, \sigma x), z)$.

  But $\delta^*(q, (\sigma x)z)$

$= \delta^*(q, \sigma(xz))$	associativity of concat.
$= \delta^*(\delta(q, \sigma), xz)$	definition of δ^*
$= \delta^*(\delta^*(\delta(q, \sigma), x), z)$	induction hypothesis
$= \delta^*(\delta^*(q, \sigma x), z)$	definition of δ^*

Proof of Claim

- $x \equiv_L y$ iff (by definition of $\equiv_L$)
- $L_x = L_y$ iff
- $\forall z \in \Sigma^*\ (xz \in L \leftrightarrow yz \in L)$ iff
- $\forall z \in \Sigma^*\ (\delta^*(q_0, xz) \in F \equiv \delta^*(q_0, yz) \in F)$, where F is the set of accepting states of M iff
- $\forall z \in \Sigma^*$
  $(\delta^*(\delta^*(q_0, x), z) \in F \leftrightarrow \delta^*(\delta^*(q_0, y), z) \in F$ iff
- $\delta^*(q_0, x) \equiv \delta^*(q_0, y)$

Impact of Claim

- The claim shows that two **strings** are equivalent iff the **states** to which the machine is taken when reading those state are also equivalent.
- But the set of states is **finite**, so the set of equivalence classes, i.e. the partition, must be finite as well.
- Hence the partition on Σ^* is also **finite**, it being in one-one correspondence with the partition on states.

Claim: If $\equiv_L$ has finite-rank, then there is a DFA accepting L.

- The DFA M is simply constructed as follows:
  - The states of M are the equivalence classes of $\equiv_L$.
  - Let [x] denote the equivalence class of $x \in \Sigma^*$.
  - The initial state of M is $[\varepsilon]$.
  - The accepting states are [x] for $x \in L$.
  - The transitions are defined by the function:
    $\forall x \in \Sigma^*\ \forall \sigma \in \Sigma\ \ \delta([x], \sigma) = [x\sigma]$
  - δ is well defined because $\equiv_L$ is a congruence relation.

Finite-State Automata

Robert M. Keller
Harvey Mudd College
March 2010

Automata

- Colloquially, an **automaton** (plural "automata") is an autonomous device (such as a robot or wind-up toy).
- In CS, the term has a more specific meaning: that of an abstract **mathematical machine** that can perform a specific function.

Uses of Automata

- There are many uses, one of which is to specify algorithms for accepting languages.
- An automaton **accepts a language** if it can tell, for any given input string, whether or not the string is in the language.

Example: Compilers, etc.

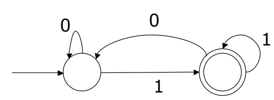
- Every compiler contains an automaton, that tells whether or not the input string is well-formed, i.e. is in the language that it compiles.
- Every pattern search program is effectively an automaton for recognizing patterns.

Finite-State Automata (FSA or DFA, they are the same)

- An automaton is finite-state if its behavior is representable by transitions between a states in a finite set, some of which are designated accepting and others not.
- Each automaton has a designated start state.

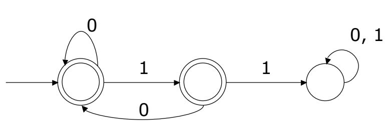
Examples of FSA

- An FSA capable of accepting exactly the strings ending with 1.



Examples of FSA

- An FSA capable of accepting exactly the strings containing no two consecutive 1's.



Thing to Check

- For each combination of a state and a symbol, there should be exactly one arrow leaving the state with that symbol.
- This is the "deterministic" ("D") in DFA.
- If this property does not hold, better fix it; your automaton might wrong.

Application

- One way to implement a search is to construct, perhaps on the fly, an automaton that accepts the corresponding language, then simulate the automaton on the given input.

Two Ways to Define Specific Languages

- Give an FSA that accepts the language.
- Give a regular expression for the language.

Remarkable Fact

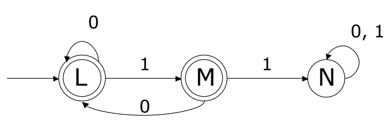
- The preceding two ways are equivalent.
- Equivalent here means that the two methods define the same family of languages.

Application of this Theory

- Sometimes it's easier to give an automaton for a language.
- Sometimes it's easier to give a regular expression.
- It would be nice to be able to go from one to the other more-or-less freely.

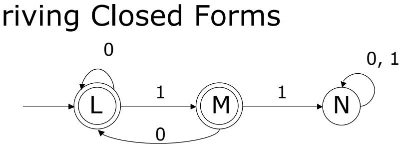
Regular Expression from DFA

- Label the States



- Identify each state with the set of paths from the start state to it. This set is a language.
- The language accepted by the DFA is the **union** of the paths to each of the accepting states, in this case L ∪ M.

Deriving Closed Forms



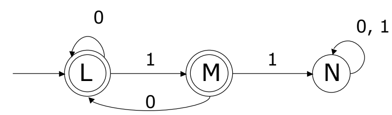
- View the acceptor as a set of **regular-expression equations**:
  - $L = L0 \cup M0 \cup \varepsilon$
  - $M = L1$
  - $N = M1 \cup N(0 \cup 1)$
- The ε is on the RHS of the **starting state** only.
- We want to solve (e.g. using Arden's Rule) for L and M, and take the union of the solutions.

Solving RE Equations

- **Solve** for L and M:
  - $L = L0 \cup M0 \cup \varepsilon$
  - $M = L1$
  - $N = M1 \cup N(0 \cup 1)$
- **Substitution** Operation:
  - A LHS variable can be replaced with its RHS, so replacing M in the L equation:
  - $L = L0 \cup L10 \cup \varepsilon$, or more simply
  - $L = L(0 \cup 10) \cup \varepsilon$
- **Elimination** Operation:
  - An equation of the form $L = LA \cup B$ has the solution $L = BA^*$, so:
  - $L = \varepsilon(0 \cup 10)^*$, or more simply $L = (0 \cup 10)^*$
- Substitution again:
  - $M = L1$
  - $M = (0 \cup 10)^*1$

Conclusion

- The language accepted by the DFA below is
  - $L \cup M$
  - which is $(0 \cup 10)^* \cup (0 \cup 10)^*1$
  - or more simply
  - $(0 \cup 10)^*(\varepsilon \cup 1)$



DFA ⇒ RE Algorithm

- Express the FSA as a set of RE equations
  - Each state is a variable.
  - Each variable is equated to a union of expressions showing how to get to that state in one step from other states.
  - The start state has ε on the RHS as well.
- Solve the RE equations for the variables:
  - The variables, along with their equations, are solved for one at a time.
  - Choose a variable for elimination.
  - Express that variable in terms of the remaining variables only, using the * operator ($L = LA \cup B$ has the solution $L = BA^*$).
  - Substitute the solution for all occurrences of the variable in the remaining equations.
  - Repeat the above steps until no variables remain.
- Work backward, substituting the solutions found for other variables, until each variable is expressed in closed form.

Another Example

- **Solve**:
  - $L = L1 \cup M0 \cup N0 \cup \varepsilon$
  - $M = L0 \cup M1 \cup N1$
  - $N = L1 \cup M1 \cup N0$
- Note that these equations don't really correspond to a deterministic machine, but it doesn't matter.
- Eliminate N, using $N = (L1 \cup M1)0^*$
  - $L = L1 \cup M0 \cup (L1 \cup M1)0^*0 \cup \varepsilon$
  - $M = L0 \cup M1 \cup (L1 \cup M1)0^*1$
- Regroup:
  - $L = L(1 \cup 10^*0) \cup M(0 \cup 10^*0) \cup \varepsilon$
  - $M = L(0 \cup 10^*1) \cup M(1 \cup 10^*1)$

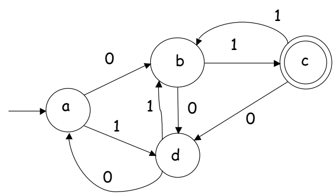
Solution, continued

- Solving:
  - $L = L(1 \cup 10^*0) \cup M(0 \cup 10^*0) \cup \varepsilon$
  - $M = L(0 \cup 10^*1) \cup M(1 \cup 10^*1)$
- Eliminate M using $M = L(0 \cup 10^*1)(1 \cup 10^*1)$, giving:
  - $L = L(1 \cup 10^*0) \cup L(0 \cup 10^*1)(1 \cup 10^*1)(0 \cup 10^*0) \cup \varepsilon$
- Regrouping:
  - $L = L((1 \cup 10^*0) \cup (0 \cup 10^*1)(1 \cup 10^*1)(0 \cup 10^*0)) \cup \varepsilon$
- Solving:
  - $L = ((1 \cup 10^*0) \cup (0 \cup 10^*1)(1 \cup 10^*1)(0 \cup 10^*0))^*$
- Working backward:
  - $M = ((1 \cup 10^*0) \cup (0 \cup 10^*1)(1 \cup 10^*1)(0 \cup 10^*0))^*$ $(0 \cup 10^*1)(1 \cup 10^*1)$
  - $N = (L1 \cup M1)0^* = \ldots$

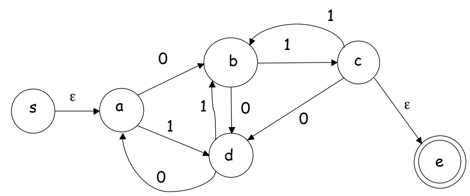
Graphical Alternative Viewpoint

- The DFA is interpreted graphically as a set of regular-expression equations.
- After an initial setup, nodes are eliminated one at a time, replacing paths through them with regular expressions.
- At completion, there is one arc between a pair of nodes, labeled with the regular expression for the language of the DFA.

DFA → RE Example



Step 1: Add Isolated Start and End States



Ultimate goal

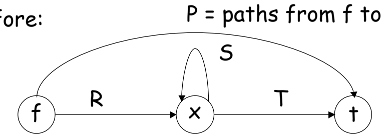
Elimination Step

- Pick a node for elimination (other than start and end).
- Union to the regular expression of **each pair** of nodes having a path through the chosen node an additional expression component representing those paths.

Elimination Step Illustrated

Before:

P = paths from f to t

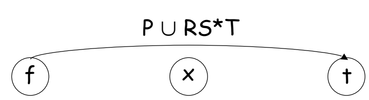


from-state eliminate to-state

added paths from f to t:

Elimination Step Illustrated

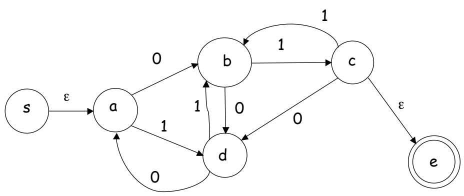
After:



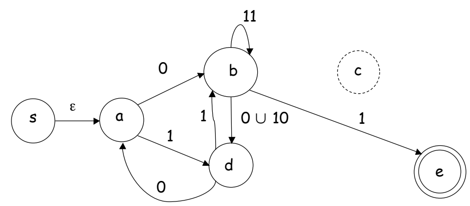
from-state eliminate to-state

This has to be done for **all** pairs f, t **including** the case where f = t.

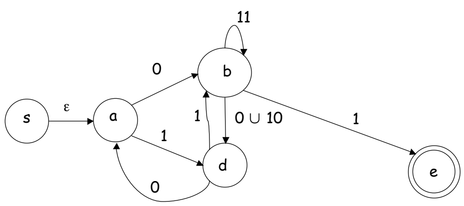
Eliminate c



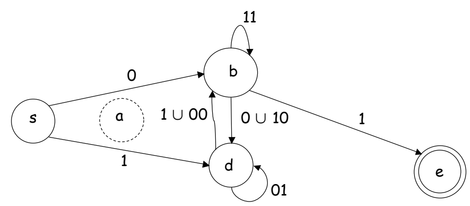
c Eliminated



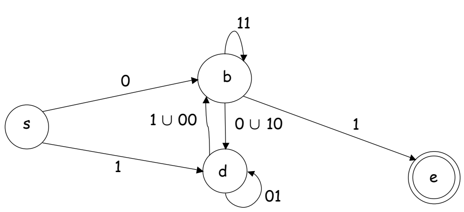
Eliminate a



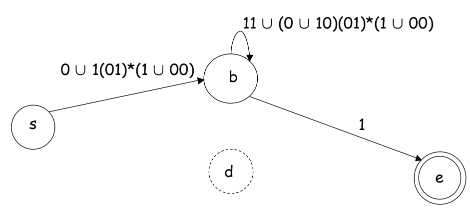
a Eliminated



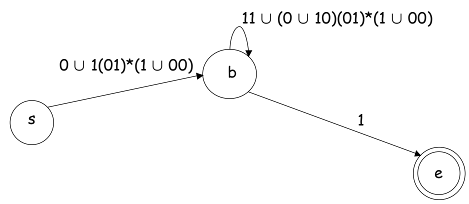
Eliminate d



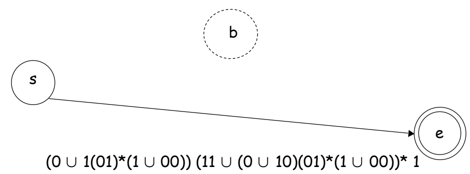
d Eliminated



Eliminate b



b Eliminated (= done)



Summary so far

- The language accepted by an DFA is a regular language.
- We haven't yet shown that the converse is true.