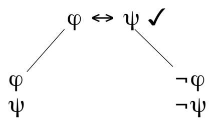
Tableau and Sequent Calculus for Predicate Logic

Robert Keller
February 2010

Aside: Additional Tree Rules: $\varphi \leftrightarrow \psi$

- This formula is retired and the tree **branches** with the formulas and their negations stacked:

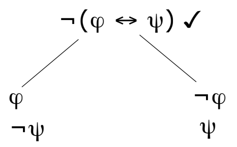

$\varphi \leftrightarrow \psi$ ✓

φ
ψ

$\neg\varphi$
$\neg\psi$

Additional Tree Rules: $\neg(\varphi \leftrightarrow \psi)$

- This formula is retired and the tree **branches**, with φ negated:


$\neg(\varphi \leftrightarrow \psi)$ ✓

φ
$\neg\psi$

$\neg\varphi$
ψ

Propositional Tree Rule Summary

stack	$\varphi \wedge \psi$ φ ψ	$\neg(\varphi \vee \psi)$ $\neg\varphi$ $\neg\psi$	$\neg(\varphi \rightarrow \psi)$ φ $\neg\psi$		$\neg\neg\varphi$ φ
split	$\varphi \vee \psi$ φ ψ	$\neg(\varphi \wedge \psi)$ $\neg\varphi$ $\neg\psi$	$\varphi \rightarrow \psi$ $\neg\varphi$ ψ	$(\varphi \leftrightarrow \psi)$ φ $\neg\varphi$ ψ $\neg\psi$ $\neg(\varphi \leftrightarrow \psi)$ φ $\neg\varphi$ $\neg\psi$ ψ	

Quantifier Rules for Tableaux

¬∃ rule

$\neg\exists v\ \varphi$ ✓
$\forall v\ \neg\varphi$

¬∀ rule

¬∀ν φ ✓
∃ν ¬φ

∃ rule

∃ν φ ✓
φ[c/ν]

where c is a **new** constant not appearing in the tree.

The rule can be used only once per ∃ formula.

∀ rule

∀ν φ **Does not get a check!!**
φ[τ/ν]

where τ is any term free to replace ν in φ.

This rule can used arbitrarily-many times for a ∀ formula.

Choice of Terms for ∀ rule

- Choose terms constructed of constants introduced by the ∃ rule

Example (already negated)

¬(∀x p(x) → ∃x p(x))

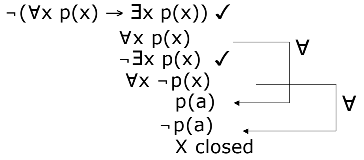
Example

¬(∀x p(x) → ∃x p(x)) ✓
∀x p(x)
¬∃x p(x)

Example

¬(∀x p(x) → ∃x p(x)) ✓
∀x p(x)
¬∃x p(x) ✓
∀x ¬p(x)

Example


¬(∀x p(x) → ∃x p(x)) ✓
∀x p(x)
¬∃x p(x) ✓
∀x ¬p(x)
p(a)
¬p(a)
X closed

The root formula is not satisfiable.
Thus ∀x p(x) → ∃x p(x) is valid

Closure depended on appropriate choice of term to substitute for x in ∀x ¬p(x).

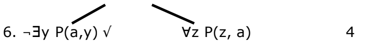
Example

(in which ∀ rule is used twice from the same line)

∀x (∃y P(x,y) → ∀z P(z, x)), P(a,a) |— P(a, b)

This time we number for better clarity.

1. ∀x (∃y P(x,y) → ∀z P(z, x))	premise
2. P(a,a)	premise
3. ¬P(a, b)	negated conclusion
4. (∃y P(a,y) → ∀z P(z, a)) √	1 with a for x
5. (∃y P(b,y) → ∀z P(z, b))	1 with b for x

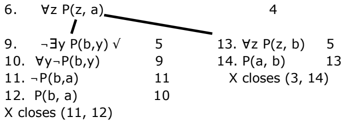


6. ¬∃y P(a,y) √	∀z P(z, a)	4
7. ∀y¬P(a, y)	(cont'd next pg)	6
8. ¬P(a, a)		7

X closes (2, 8)

Example continued

1. ∀x (∃y P(x,y) → ∀z P(z, x))	premise
2. P(a,a)	premise
3. ¬P(a, b)	negated conclusion
4. (∃y P(a,y) → ∀z P(z, a)) √	1 with a for x
5. (∃y P(b,y) → ∀z P(z, b)) √	1 with b for x
6. ∀z P(z, a)	4



9. ¬∃y P(b,y) √	5	13. ∀z P(z, b)	5
10. ∀y¬P(b,y)	9	14. P(a, b)	13
11. ¬P(b,a)	11	X closes (3, 14)	
12. P(b, a)	10		
X closes (11, 12)			

Termination

- Unlike the propositional case, the predicate version of tableaux does not necessarily terminate. This is because the ∀ rule can be used arbitrarily-many times.
- It can be shown, however, that **if** the root formula is unsatisfiable, then **there exists** a closed tree for it.
  (If the root formula is satisfiable, the construction *might* not terminate.)

Example of Non-Termination

∀x ∃y P(x, y) |— P(a, a)

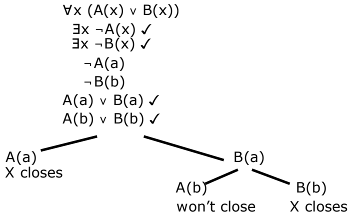
1. ∀x ∃y P(x, y)	premise
2. ¬ P(a, a)	conclusion negated
3. ∃y P(a, y) √	1, a for x
4. P(a, b)	3, b for y
5. ∃y P(b, y) √	1, b for x
6. P(b, c)	5, c for y
7. ∃y P(c, y) √	1, c for x
8. P(c, d)	7, d for y
9. ∃y P(d, y) √	1, d for x
. . .	. . .

Using the Tableau Method to Find a Model in the Predicate Calculus

- ∀x (A(x) ∨ B(x)) → (∀x A(x) ∨ ∀x B(x))
- This formula is not valid, so its negation should be satisfiable.

¬(∀x (A(x) ∨ B(x)) → (∀x A(x) ∨ ∀x B(x))) ✓
∀x (A(x) ∨ B(x))
¬(∀x A(x) ∨ ∀x B(x)) ✓
¬∀x A(x) ✓
¬∀x B(x) ✓
∃x ¬A(x)
∃x ¬B(x)

Using the Tableau Method to Find a Model in the Predicate Calculus


∀x (A(x) ∨ B(x))
∃x ¬A(x) ✓
∃x ¬B(x) ✓
¬A(a)
¬B(b)
A(a) ∨ B(a) ✓
A(b) ∨ B(b) ✓
A(a)
X closes
B(a)
A(b)
won’t close
B(b)
X closes

Conclusion: **There is a model for the negation** with domain {a, b}, in which “¬A(a), A(b), B(a), and ¬B(b)” (translated into the appropriate interpretation notation).

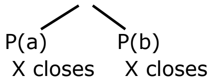
Handling Equality in Tableaux

- If an open path has a node $t_1 = t_2$, then for any unchecked node containing t_1, add on the path a formula in which t_1 is replaced with t_2 (and vice-versa).

Example: Equality in Tableau

- P(a)∨P(b), ¬P(a) |— ¬(a=b)
- Proof:

1. P(a)∨P(b) premise
2. ¬P(a) premise
3. ¬¬(a=b) ✓ negated conclusion
4. a=b 3, ¬¬
5. ¬P(b) 2, 4, = rule


P(a)
X closes
P(b)
X closes
1

Example: Equality in Tableau

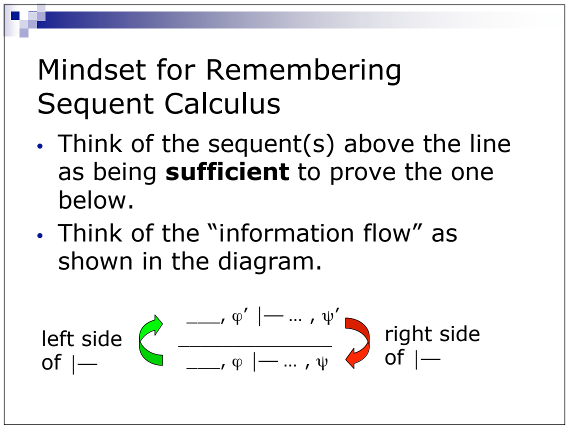
- a = b |— P(a, b) → P(b, a)
- Proof:

1. a =b premise
2. ¬(P(a, b) → P(b, a)) ✓ negated conclusion
3. P(a, b) 2
4. ¬P(b, a) 2
5. P(a, a) 3, 1, = rule
6. ¬P(a, a) 4, 1, = rule

X closes(5, 6)

Sequent Calculus for Predicates

- As with the tableau method, the propositional rules for Sequent Calculus will be augmented with four new rules for quantifiers.
- As before, the Sequent Calculus rules have a correspondence with the tableau proof rules.
- Whereas the tableau shows negation explicitly, in Sequent Calculus it may be implicit, depending on which side of the turnstile a formula appears.
- Negated formulas in tableaux generally correspond to formulas on the right of the turnstile in Sequent Calculus.



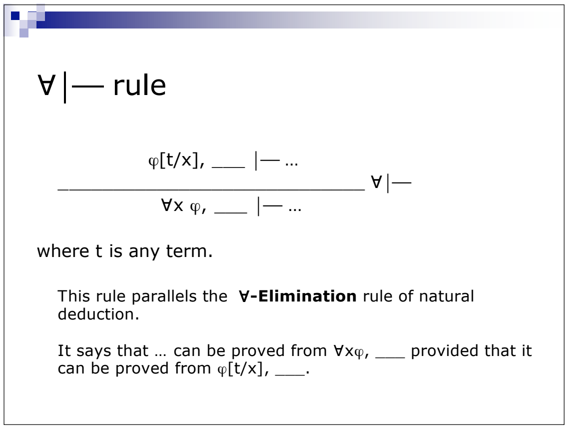
Mindset for Remembering Sequent Calculus

- Think of the sequent(s) above the line as being **sufficient** to prove the one below.
- Think of the "information flow" as shown in the diagram.

left side of |—

$$\frac{\_\_\_,\ \varphi' \vdash \ldots,\ \psi'}{\_\_\_,\ \varphi \vdash \ldots,\ \psi}$$

right side of |—



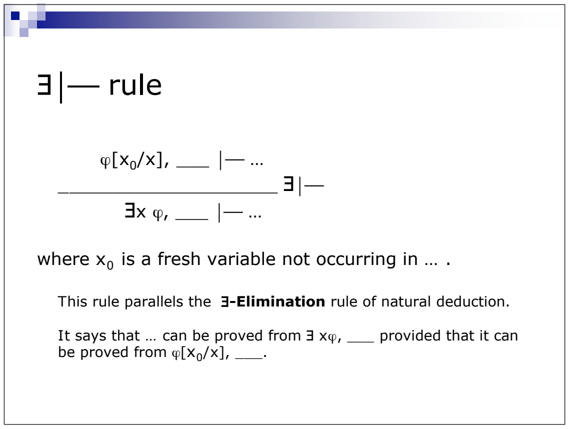
∀|— rule

$$\frac{\varphi[t/x],\ \_\_\_\ \vdash \ldots}{\forall x\ \varphi,\ \_\_\_\ \vdash \ldots}\ \forall\vdash$$

where t is any term.

This rule parallels the **∀-Elimination** rule of natural deduction.

It says that … can be proved from ∀xφ, ___ provided that it can be proved from φ[t/x], ___.



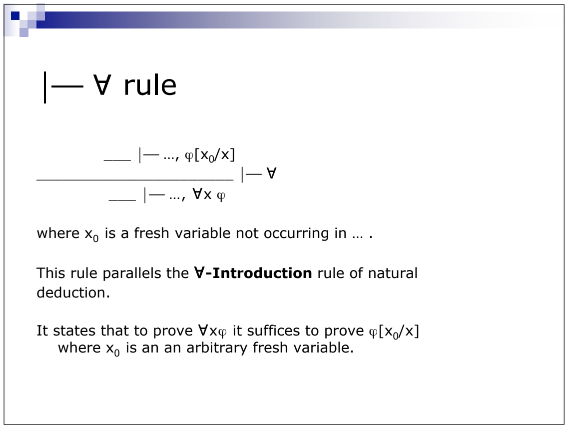
∃|— rule

$$\frac{\varphi[x_0/x],\ \_\_\_\ \vdash \ldots}{\exists x\ \varphi,\ \_\_\_\ \vdash \ldots}\ \exists\vdash$$

where x_0 is a fresh variable not occurring in … .

This rule parallels the **∃-Elimination** rule of natural deduction.

It says that … can be proved from ∃ xφ, ___ provided that it can be proved from $\varphi[x_0/x]$, ___.



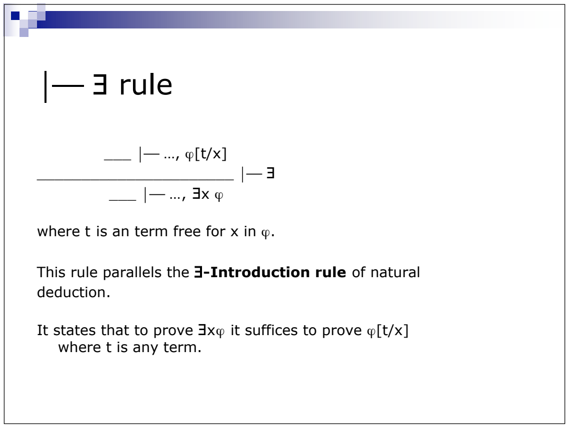
|— ∀ rule

$$\frac{\_\_\_\ \vdash \ldots,\ \varphi[x_0/x]}{\_\_\_\ \vdash \ldots,\ \forall x\ \varphi}\ \vdash\forall$$

where x_0 is a fresh variable not occurring in … .

This rule parallels the **∀-Introduction** rule of natural deduction.

It states that to prove ∀xφ it suffices to prove $\varphi[x_0/x]$ where x_0 is an an arbitrary fresh variable.



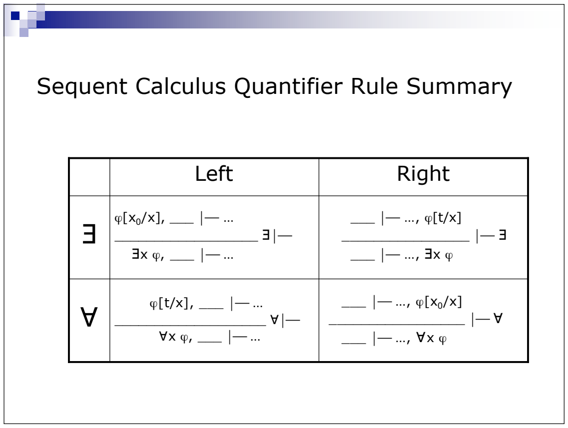
|— ∃ rule

$$\frac{\_\_\_\ \vdash \ldots,\ \varphi[t/x]}{\_\_\_\ \vdash \ldots,\ \exists x\ \varphi}\ \vdash\exists$$

where t is an term free for x in φ.

This rule parallels the **∃-Introduction rule** of natural deduction.

It states that to prove ∃xφ it suffices to prove φ[t/x] where t is any term.

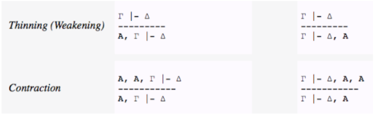


Sequent Calculus Quantifier Rule Summary

	Left	Right
∃	$\frac{\varphi[x_0/x],\ \_\_\_\ \vdash \ldots}{\exists x\ \varphi,\ \_\_\_\ \vdash \ldots}\ \exists\vdash$	$\frac{\_\_\_\ \vdash \ldots,\ \varphi[t/x]}{\_\_\_\ \vdash \ldots,\ \exists x\ \varphi}\ \vdash\exists$
∀	$\frac{\varphi[t/x],\ \_\_\_\ \vdash \ldots}{\forall x\ \varphi,\ \_\_\_\ \vdash \ldots}\ \forall\vdash$	$\frac{\_\_\_\ \vdash \ldots,\ \varphi[x_0/x]}{\_\_\_\ \vdash \ldots,\ \forall x\ \varphi}\ \vdash\forall$

Variations

- See the tutorial by Alexander Sakharov: http://sakharov.net/sequent.html
- In particular, some versions uses the ability to “thin” and “contract” a set of formulas.
- We may use these on occasion.



Thinning (Weakening)	Γ |- Δ --------- A, Γ |- Δ	Γ |- Δ --------- Γ |- Δ, A
Contraction	A, A, Γ |- Δ ----------- A, Γ |- Δ	Γ |- Δ, A, A ----------- Γ |- Δ, A

Rule Correspondence

Tableau Rule	Sequent Calculus Rule (going backward/upward)
¬∃ replace with ∀¬	|— ∃ specializes term on right
¬∀ replace with ∃¬	|— ∀ specializes term on right to fresh var
∃ introduces constant	∃|— specializes term on left to fresh var
∀ uses term	∀|— specializes term on left

Sequent Calculus Examples

∀x p(x) → ∃x p(x) revisited
using JAPE

$$(\forall x.P(x))\to(\exists x.P(x))$$

Sequent Calculus Examples

|— → rule

$$\dfrac{\forall x.P(x) \vdash \exists x.P(x)}{(\forall x.P(x))\to(\exists x.P(x))}\ \vdash\to$$

Sequent Calculus Examples

- |— ∃ rule (introduces term _B for x)

$$\dfrac{\dfrac{\forall x.P(x) \vdash P(\_B)}{\forall x.P(x) \vdash \exists x.P(x)}\ \vdash\exists}{(\forall x.P(x))\to(\exists x.P(x))}\ \vdash\to$$

Sequent Calculus Examples

unifying _B with term x (all that is available!?)

Note this x is distinct from the one in ∀x.P(x).

$$\dfrac{\dfrac{\forall x.P(x) \vdash P(x)}{\forall x.P(x) \vdash \exists x.P(x)}\ \vdash\exists}{(\forall x.P(x))\to(\exists x.P(x))}\ \vdash\to$$

Sequent Calculus Examples

∀|— rule

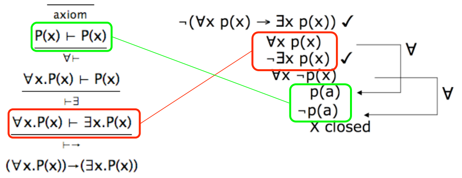
$$\dfrac{\dfrac{\dfrac{P(\_B) \vdash P(x)}{\forall x.P(x) \vdash P(x)}\ \forall\vdash}{\forall x.P(x) \vdash \exists x.P(x)}\ \vdash\exists}{(\forall x.P(x))\to(\exists x.P(x))}\ \vdash\to$$

Sequent Calculus Examples

Unifying _B with x gives an axiom

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P(x) \vdash P(x)}\ \text{axiom}}{\forall x.P(x) \vdash P(x)}\ \forall\vdash}{\forall x.P(x) \vdash \exists x.P(x)}\ \vdash\exists}{(\forall x.P(x))\to(\exists x.P(x))}\ \vdash\to$$

Compare with Tableau



$$\dfrac{\dfrac{\dfrac{\dfrac{}{P(x) \vdash P(x)}\ \text{axiom}}{\forall x.P(x) \vdash P(x)}\ \forall\vdash}{\forall x.P(x) \vdash \exists x.P(x)}\ \vdash\exists}{(\forall x.P(x))\to(\exists x.P(x))}\ \vdash\to$$

¬(∀x p(x) → ∃x p(x)) ✓
∀x p(x)
¬∃x p(x) ✓
∀x ¬p(x)
p(a)
¬p(a)
X closed

As before, could use block tableau to make the connection clearer.

Another Sequent Proof Example (Here contraction will be used.)

The LHS is just a way to get and individual. (This theory in JAPE does not have a "top" T.)

$$\exists x.B(x) \vdash \exists y.(A(y)\to(\forall z.A(z)))$$

Another Sequent Proof Example

m is a fresh variable
(essentially this is ∃ elim.)

$$\dfrac{B(m) \vdash \exists y.(A(y)\to(\forall z.A(z)))}{\exists x.B(x) \vdash \exists y.(A(y)\to(\forall z.A(z)))}\ \exists\vdash$$

Another Sequent Proof Example

(Here contraction is used to keep a copy of a formula in the RHS for later.

This observation was a result of failing to complete the proof without the copy.)

$$\dfrac{\dfrac{B(m) \vdash \begin{array}{l}\exists y.(A(y)\to(\forall z.A(z))),\\ \exists y.(A(y)\to(\forall z.A(z)))\end{array}}{B(m) \vdash \exists y.(A(y)\to(\forall z.A(z)))}\ \vdash\text{contract}}{\exists x.B(x) \vdash \exists y.(A(y)\to(\forall z.A(z)))}\ \exists\vdash$$

Another Sequent Proof Example

Need a term for _B1.
(Essentially this is ∃ intro.)

$$\begin{array}{c}
B(m) \vdash A(\_B1)\to(\forall z.A(z)),\ \exists y.(A(y)\to(\forall z.A(z))) \\ \hline
{\scriptstyle \vdash\exists} \\
B(m) \vdash \exists y.(A(y)\to(\forall z.A(z))),\ \exists y.(A(y)\to(\forall z.A(z))) \\ \hline
{\scriptstyle \vdash\text{contract}} \\
B(m) \vdash \exists y.(A(y)\to(\forall z.A(z))) \\ \hline
{\scriptstyle \exists\vdash} \\
\exists x.B(x) \vdash \exists y.(A(y)\to(\forall z.A(z)))
\end{array}$$

Another Sequent Proof Example

Unify _B with m:

$$\begin{array}{c}
B(m) \vdash A(m)\to(\forall z.A(z)),\ \exists y.(A(y)\to(\forall z.A(z))) \\ \hline
{\scriptstyle \vdash\exists} \\
B(m) \vdash \exists y.(A(y)\to(\forall z.A(z))),\ \exists y.(A(y)\to(\forall z.A(z))) \\ \hline
{\scriptstyle \vdash\text{contract}} \\
B(m) \vdash \exists y.(A(y)\to(\forall z.A(z))) \\ \hline
{\scriptstyle \exists\vdash} \\
\exists x.B(x) \vdash \exists y.(A(y)\to(\forall z.A(z)))
\end{array}$$

Another Sequent Proof Example

(Essentially want ∀ intro.)
Need a fresh var for z.

$$\begin{array}{c}
B(m),\ A(m) \vdash \forall z.A(z),\ \exists y.(A(y)\to(\forall z.A(z))) \\ \hline
{\scriptstyle \vdash\to} \\
B(m) \vdash A(m)\to(\forall z.A(z)),\ \exists y.(A(y)\to(\forall z.A(z))) \\ \hline
{\scriptstyle \vdash\exists} \\
B(m) \vdash \exists y.(A(y)\to(\forall z.A(z))),\ \exists y.(A(y)\to(\forall z.A(z))) \\ \hline
{\scriptstyle \vdash\text{contract}} \\
B(m) \vdash \exists y.(A(y)\to(\forall z.A(z))) \\ \hline
{\scriptstyle \exists\vdash} \\
\exists x.B(x) \vdash \exists y.(A(y)\to(\forall z.A(z)))
\end{array}$$

Another Sequent Proof Example

m1 is a fresh var for z.
Now use ∃ intro.

$$\begin{array}{c}
B(m),\ A(m) \vdash A(m1),\ \exists y.(A(y)\to(\forall z.A(z))) \\ \hline
{\scriptstyle \vdash\forall} \\
B(m),\ A(m) \vdash \forall z.A(z),\ \exists y.(A(y)\to(\forall z.A(z))) \\ \hline
{\scriptstyle \vdash\to} \\
B(m) \vdash A(m)\to(\forall z.A(z)),\ \exists y.(A(y)\to(\forall z.A(z))) \\ \hline
{\scriptstyle \vdash\exists} \\
B(m) \vdash \exists y.(A(y)\to(\forall z.A(z))),\ \exists y.(A(y)\to(\forall z.A(z))) \\ \hline
{\scriptstyle \vdash\text{contract}} \\
B(m) \vdash \exists y.(A(y)\to(\forall z.A(z))) \\ \hline
{\scriptstyle \exists\vdash} \\
\exists x.B(x) \vdash \exists y.(A(y)\to(\forall z.A(z)))
\end{array}$$

Another Sequent Proof Example

The payoff for the earlier contraction step:

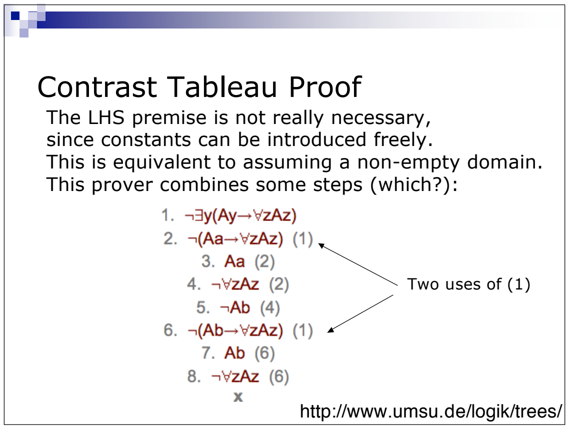
$$\begin{array}{c}
B(m),\ A(m) \vdash A(\_B1)\to(\forall z.A(z)),\ A(m1) \\ \hline
{\scriptstyle \vdash\exists} \\
B(m),\ A(m) \vdash A(m1),\ \exists y.(A(y)\to(\forall z.A(z))) \\ \hline
{\scriptstyle \vdash\forall} \\
B(m),\ A(m) \vdash \forall z.A(z),\ \exists y.(A(y)\to(\forall z.A(z))) \\ \hline
{\scriptstyle \vdash\to} \\
B(m) \vdash A(m)\to(\forall z.A(z)),\ \exists y.(A(y)\to(\forall z.A(z))) \\ \hline
{\scriptstyle \vdash\exists} \\
B(m) \vdash \exists y.(A(y)\to(\forall z.A(z))),\ \exists y.(A(y)\to(\forall z.A(z))) \\ \hline
{\scriptstyle \vdash\text{contract}} \\
B(m) \vdash \exists y.(A(y)\to(\forall z.A(z))) \\ \hline
{\scriptstyle \exists\vdash} \\
\exists x.B(x) \vdash \exists y.(A(y)\to(\forall z.A(z)))
\end{array}$$

unify _B1, m1:

$$\begin{array}{c}
B(m),\ A(m) \vdash A(m1)\to(\forall z.A(z)),\ A(m1) \\ \hline
{\scriptstyle \vdash\exists} \\
B(m),\ A(m) \vdash A(m1),\ \exists y.(A(y)\to(\forall z.A(z))) \\ \hline
{\scriptstyle \vdash\forall} \\
B(m),\ A(m) \vdash \forall z.A(z),\ \exists y.(A(y)\to(\forall z.A(z))) \\ \hline
{\scriptstyle \vdash\to} \\
B(m) \vdash A(m)\to(\forall z.A(z)),\ \exists y.(A(y)\to(\forall z.A(z))) \\ \hline
{\scriptstyle \vdash\exists} \\
B(m) \vdash \exists y.(A(y)\to(\forall z.A(z))),\ \exists y.(A(y)\to(\forall z.A(z))) \\ \hline
{\scriptstyle \vdash\text{contract}} \\
B(m) \vdash \exists y.(A(y)\to(\forall z.A(z))) \\ \hline
{\scriptstyle \exists\vdash} \\
\exists x.B(x) \vdash \exists y.(A(y)\to(\forall z.A(z)))
\end{array}$$

Proof Concluded

$$\begin{array}{c}
\hline
{\scriptstyle \text{axiom}} \\
B(m),\ A(m),\ A(m1) \vdash \forall z.A(z),\ A(m1) \\ \hline
{\scriptstyle \vdash\to} \\
B(m),\ A(m) \vdash A(m1)\to(\forall z.A(z)),\ A(m1) \\ \hline
{\scriptstyle \vdash\exists} \\
B(m),\ A(m) \vdash A(m1),\ \exists y.(A(y)\to(\forall z.A(z))) \\ \hline
{\scriptstyle \vdash\forall} \\
B(m),\ A(m) \vdash \forall z.A(z),\ \exists y.(A(y)\to(\forall z.A(z))) \\ \hline
{\scriptstyle \vdash\to} \\
B(m) \vdash A(m)\to(\forall z.A(z)),\ \exists y.(A(y)\to(\forall z.A(z))) \\ \hline
{\scriptstyle \vdash\exists} \\
B(m) \vdash \exists y.(A(y)\to(\forall z.A(z))),\ \exists y.(A(y)\to(\forall z.A(z))) \\ \hline
{\scriptstyle \vdash\text{contract}} \\
B(m) \vdash \exists y.(A(y)\to(\forall z.A(z))) \\ \hline
{\scriptstyle \exists\vdash} \\
\exists x.B(x) \vdash \exists y.(A(y)\to(\forall z.A(z)))
\end{array}$$



Contrast Tableau Proof

The LHS premise is not really necessary,
since constants can be introduced freely.
This is equivalent to assuming a non-empty domain.
This prover combines some steps (which?):

1. ¬∃y(Ay→∀zAz)
2. ¬(Aa→∀zAz) (1)
3. Aa (2)
4. ¬∀zAz (2)
5. ¬Ab (4)
6. ¬(Ab→∀zAz) (1)
7. Ab (6)
8. ¬∀zAz (6)

x

Two uses of (1)

http://www.umsu.de/logik/trees/