(Imperative) Program Logic Part 2

Robert Keller

March 2010

Inferring invariants

- There is no fully general automation for inferring invariants (as there is for the weakest precondition of assignment statements).
- This is one of the things that makes totally automated verification difficult.
- Finding the right invariant is still a human intellectual activity.

Approaching Invariants through the "Back Door"

- Suppose A is an assertion that we desire to be true after a while loop.
- Suppose the invariant I is unknown.
- We do know that whatever I is, it satisfies:

$$(I \wedge \neg P) \rightarrow A$$

by comparing the template:

$\{I\}$ **while P do B** $\{I \wedge \neg P\}$

$\{I\}$ **while P do B** $\{A\}$

Approximating I

- For starters, then, we could "over−approximate" I by A.

- Then working backward through the body B, we derive the "weakest pre-condition" corresponding to A, denoted:
  $wp_B(A)$

- We also know that
  $(I \wedge \neg P) \rightarrow wp_B(A)$
  so we can use $wp_B(A)$ as another approximation to I.

- Continuing, I is approximated by
  $A \vee wp_B(A) \vee wp_B(wp_B(A)) \vee wp_B(wp_B(wp_B(A))) \vee \ldots$

Example

-
```
i := 0;
while i < n
      do
      i := i+1
      od
{i = n}
```

$wp_{i\ :=\ i+1}(i = n)$ is $i+1 = n$ which is the same as $i = n-1$.
$wp_{i\ :=\ i+1}(wp_{i\ :=\ i+1}(i = n))$ is $i+1 = n-1$ which is $i = n-2$

So the continued approximation is:
$(i = n) \vee (i = n-1) \vee (i = n-2) \vee \ldots$

suggesting as the invariant $i \leq n$, which we know works.

Moreover, for $n \geq 0$, $i \leq n$ is implied by $i = 0$, which is also required.

Proof Using JAPE

```
WHERE DISTINCT i, n IS
{0≤n}

(i := 0)

{i≤n}

while i < n
  do i := i+1 od

{i = n}
```

Proof Steps

...

1: $\{0 \le n\}(i:=0)\{i \le n\}$while $i<n$ do $i:=i+1$ od$\{i=n\}$

nTuple rule

...

1: $\{0 \le n\}(i:=0)\{i \le n\}$

Initialization

...

2: $\{i \le n\}$while $i<n$ do $i:=i+1$ od$\{i=n\}$

while loop

3: $\{0 \le n\}(i:=0)\{i \le n\}$while $i<n$ do $i:=i+1$ od$\{i=n\}$ Ntuple 1,2

Proof of Initialization

1: $\{0 \le n\}(i:=0)\{i \le n\}$ variable-assignment

...

2: $\{i \le n\}$while $i<n$ do $i:=i+1$ od$\{i=n\}$

3: $\{0 \le n\}(i:=0)\{i \le n\}$while $i<n$ do $i:=i+1$ od$\{i=n\}$ Ntuple 1,2

Expand the while

...

body

2: $\{i \le n \wedge i < n\}(i := i+1)\{i \le n\}$ Body partial correctness

...

3: $i \le n \wedge i < n \rightarrow \_M > 0$ Variant continuation condition

4: integer Km assumption

...

body

5: $\{i \le n \wedge i < n \wedge \_M = Km\}(i := i+1)\{\_M < Km\}$ Body decreases variant

6: $\{i \le n\}$while $i < n$ do $i := i+1$ od$\{i \le n \wedge \neg(i < n)\}$ while 2,3,4-5

...

7: $i \le n \wedge \neg(i < n) \rightarrow i = n$ loop exit implies final expectation

8: $\{i \le n\}$while $i < n$ do $i := i+1$ od$\{i = n\}$ consequence(R) 6,7

Body Partial Correctness

2:	$\boxed{i \leq n \wedge i < n}$	assumption
3:	$i < n$	$\wedge$ elim 2
4:	$i+1 \leq n$	$A+1 \leq B \triangleq A < B$ 3
5:	$i \leq n \wedge i < n \rightarrow i+1 \leq n$	$\rightarrow$ intro 2–4
6:	$\{i+1 \leq n\}(i := i+1)\{i \leq n\}$	variable-assignment

body

Set the Variant: n-i

...

8: $i \le n \wedge i < n \rightarrow \boxed{n - i} > 0$

9: integer Km — assumption

...

body

10: $\{i \le n \wedge i < n \wedge \boxed{n - i} = Km\}(i := i + 1)\{\boxed{n - i} < Km\}$

11: $\{i \le n\}$while $i < n$ do $i := i + 1$ od$\{i \le n \wedge \neg(i < n)\}$ — while 7,8,9–10

Prove the Variant Continuation Condition

8:	$\boxed{i \leq n \wedge i < n}$	assumption
9:	$i < n$	$\wedge$ elim 8
10:	$n - i > 0$	$i<n \vdash n-i>0$ 9
11:	$i \leq n \wedge i < n \rightarrow n - i > 0$	$\rightarrow$ intro 8–10

Prove Body Decreases Variant

12:	integer Km	assumption
13:	$i \le n \wedge i < n \wedge n-i=Km$	assumption
14:	$n-i=Km$	$\wedge$ elim 13
15:	$n-(i+1)<Km$	$n-i=X \vdash n-(i+1)<X$ 14
16:	$i \le n \wedge i < n \wedge n-i=Km \rightarrow n-(i+1)<Km$	$\rightarrow$ intro 13-15
17:	$\{n-(i+1)<Km\}(i:=i+1)\{n-i<Km\}$	variable-assignment
18:	$\{i \le n \wedge i < n \wedge \boxed{n-i=Km}\}\overset{\text{body}}{(i:=i+1)}\{\boxed{n-i<Km}\}$	consequence(L) 16,17
19:	$\{i \le n\}\text{while } i<n \text{ do } i:=i+1 \text{ od}\{i \le n \wedge \neg(i<n)\}$	while 7,11,12-18

Loop exit implies final expectation

20:	$i \le n \land \neg(i < n)$	assumption
21:	$i \le n$	$\land$ elim 20
22:	$\neg(i < n)$	$\land$ elim 20
23:	$i = n$	$i \le n,\ \neg(i < n) \vdash i = n$ 21,22
24:	$i \le n \land \neg(i < n) \rightarrow i = n$	$\rightarrow$ intro 20-23

Completed Proof

Can you identify all the parts?

1:	$\{0\le n\}(i:=0)\{i\le n\}$	variable-assignment
2:	$i\le n\wedge i<n$	assumption
3:	$i<n$	$\wedge$ elim 2
4:	$i+1\le n$	$A+1\le B\triangleq A<B$ 3
5:	$i\le n\wedge i<n\rightarrow i+1\le n$	$\rightarrow$ intro 2–4
6:	$\{i+1\le n\}(i:=i+1)\{i\le n\}$	variable-assignment
7:	$\{i\le n\wedge i<n\}(i:=i+1)\{i\le n\}$	consequence(L) 5,6
8:	$i\le n\wedge i<n$	assumption
9:	$i<n$	$\wedge$ elim 8
10:	$n-i>0$	$i<n \vdash n-i>0$ 9
11:	$i\le n\wedge i<n\rightarrow n-i>0$	$\rightarrow$ intro 8–10
12:	integer Km	assumption
13:	$i\le n\wedge i<n\wedge n-i=Km$	assumption
14:	$n-i=Km$	$\wedge$ elim 13
15:	$n-(i+1)<Km$	$n-i=X \vdash n-(i+1)<X$ 14
16:	$i\le n\wedge i<n\wedge n-i=Km\rightarrow n-(i+1)<Km$	$\rightarrow$ intro 13–15
17:	$\{n-(i+1)<Km\}(i:=i+1)\{n-i<Km\}$	variable-assignment
18:	$\{i\le n\wedge i<n\wedge n-i=Km\}(i:=i+1)\{n-i<Km\}$	consequence(L) 16,17
19:	$\{i\le n\}$while $i<n$ do $i:=i+1$ od$\{i\le n\wedge\neg(i<n)\}$	while 7,11,12–18
20:	$i\le n\wedge\neg(i<n)$	assumption
21:	$i\le n$	$\wedge$ elim 20
22:	$\neg(i<n)$	$\wedge$ elim 20
23:	$i=n$	$i\le n, \neg(i<n) \vdash i=n$ 21,22
24:	$i\le n\wedge\neg(i<n)\rightarrow i=n$	$\rightarrow$ intro 20–23
25:	$\{i\le n\}$while $i<n$ do $i:=i+1$ od$\{i=n\}$	consequence(R) 19,24
26:	$\{0\le n\}(i:=0)\{i\le n\}$while $i<n$ do $i:=i+1$ od$\{i=n\}$	Ntuple 1,25

Completed Proof

1:	$\{0 \le n\}(i := 0)\{i \le n\}$	variable-assignment	Initialization triple
2:	$i \le n \wedge i < n$	assumption	Body partial correctness
3:	$i < n$	$\wedge$ elim 2	
4:	$i+1 \le n$	$A+1 \le B \triangleq A < B$ 3	
5:	$i \le n \wedge i < n \rightarrow i+1 \le n$	$\rightarrow$ intro 2–4	
6:	$\{i+1 \le n\}(i := i+1)\{i \le n\}$	variable-assignment	
7:	$\{i \le n \wedge i < n\}(i := i+1)\{i \le n\}$	consequence(L) 5,6	
8:	$i \le n \wedge i < n$	assumption	Variant continuation condition
9:	$i < n$	$\wedge$ elim 8	
10:	$n-i > 0$	$i < n \vdash n-i > 0$ 9	
11:	$i \le n \wedge i < n \rightarrow n-i > 0$	$\rightarrow$ intro 8–10	
12:	integer Km	assumption	Body decreases variant
13:	$i \le n \wedge i < n \wedge n-i = Km$	assumption	
14:	$n-i = Km$	$\wedge$ elim 13	
15:	$n-(i+1) < Km$	$n-i = X \vdash n-(i+1) < X$ 14	
16:	$i \le n \wedge i < n \wedge n-i = Km \rightarrow n-(i+1) < Km$	$\rightarrow$ intro 13–15	
17:	$\{n-(i+1) < Km\}(i := i+1)\{n-i < Km\}$	variable-assignment	
18:	$\{i \le n \wedge i < n \wedge n-i = Km\}(i := i+1)\{n-i < Km\}$	consequence(L) 16,17	
19:	$\{i \le n\}$while $i < n$ do $i := i+1$ od$\{i \le n \wedge \neg(i < n)\}$	while 7,11,12–18	while triple
20:	$i \le n \wedge \neg(i < n)$	assumption	while exit implies final expectation
21:	$i \le n$	$\wedge$ elim 20	
22:	$\neg(i < n)$	$\wedge$ elim 20	
23:	$i = n$	$i \le n,\ \neg(i < n) \vdash i = n$ 21,22	
24:	$i \le n \wedge \neg(i < n) \rightarrow i = n$	$\rightarrow$ intro 20–23	
25:	$\{i \le n\}$while $i < n$ do $i := i+1$ od$\{i = n\}$	consequence(R) 19,24	while triple + consequent
26:	$\{0 \le n\}(i := 0)\{i \le n\}$while $i < n$ do $i := i+1$ od$\{i = n\}$	Ntuple 1,25	program nTuple

Proof as Trees

Partial Correctness Tree

assign

$$7: \{i \le n \wedge i < n\}(i := i+1)\{i \le n\}$$

pure logic

$$24: i \le n \wedge \neg(i<n) \rightarrow i=n$$

$$19: \{i \le n\} \text{while } i<n \text{ do } i := i+1 \text{ od}\{i \le n \wedge \neg(i<n)\}$$

assign

$$1: \{0 \le n\}(i := 0)\{i \le n\}$$

$$25: \{i \le n\} \text{while } i<n \text{ do } i := i+1 \text{ od}\{i=n\}$$

$$26: \{0 \le n\}(i := 0)\{i \le n\} \text{while } i<n \text{ do } i := i+1 \text{ od}\{i=n\}$$

Termination Tree

pure logic

$$16: i \le n \wedge i < n \wedge n-i = Km \rightarrow n-(i+1) < Km$$

assign

$$17: \{n-(i+1) < Km\}(i := i+1)\{n-i < Km\}$$

pure logic

$$11: i \le n \wedge i < n \boxed{\rightarrow n-i >} 0$$

$$18: \{i \le n \wedge i < n \boxed{\wedge n-i =} Km\}(i := i+1)\boxed{\{n-i <} Km\}$$

$$19: \{i \le n\} \text{while } i<n \text{ do } i := i+1 \text{ od}\{i \le n \wedge \neg(i<n)\}$$

Subtleties About Loop Invariants

- Can the following Ntuple be proved?

```
{0≤n}

(j := 1; i := 0; y := 0)

{y=i×i ∧ 0≤i ∧ i≤n}

while i < n
  do
       y := y+j;
       j  := j+2;
       i  := i+1
 od

{y=n×n}
```

Is the invariant truly invariant?

Loop Body Correctness Proof Reduces to

11: $y = i \times i \wedge 0 \le i \wedge i \le n \wedge i < n$

$\ldots$

12: $y + j = (i+1) \times (i+1) \wedge 0 \le i+1 \wedge i+1 \le n$

This cannot be proved.
j is not even mentioned in the hypothesis.

What went wrong?

Important: Strength of Invariants

- Our invariant was “too weak” to enable its own inductive proof.

- In making an invariant stronger, we get more information from which to derive the post condition.

- At the same time, however, we have **more to prove** with a stronger invariant, so we don’t want to go too far.

The Lattice of Invariants

$\top$ very weak, implies nothing but itself, but implied by everything

$\uparrow$

I' other invariants, implied by I^*, but too weak in themselves to enable inductive proof

$\uparrow$

I^* an **inductive** invariant, “just right”, but not necessarily unique

$\uparrow$ implies

$\perp$ very strong, implies everything, but implied by nothing but itself

An Inductive Invariant

$\{0 \le n\}$

(j := 1; i := 0; y := 0)

$\{y = i \times i \wedge \boxed{j = 2 \times i} \wedge 0 \le i \wedge i \le n\}$

```
while i < n
  do
          y := y+j;
          j  := j+2;
          i  := i+1
 od
```

$\{y = n \times n\}$

Now the Body Proof Works

14:	$y=i\times i \wedge \boxed{j=2\times i+1 \wedge} 0\le i \wedge i\le n \wedge i<n$	assumption
15:	$y=i\times i$	$\wedge$ elim 14
16:	$j=2\times i+1$	$\wedge$ elim 14
17:	$0\le i$	$\wedge$ elim 14
18:	$i<n$	$\wedge$ elim 14
19:	$y+j=(i+1)\times(i+1)$	$y=i\times i,\ j=2\times i+1 \vdash$ y... 15,16
20:	$j+2=2\times(i+1)+1$	$j=2\times i+1 \vdash j+2=2\times($i+... 16
21:	$0\le i+1$	$A\le B \vdash A\le B+1$ 17
22:	$i+1\le n$	$A<N \vdash A+1\le N$ 18
23:	$y+j=(i+1)\times(i+1)\wedge j+2=2\times(i+1)+1 \wedge 0\le i+1 \wedge i+1\le n$	$\wedge$ intro 19,20,21,22

The Completed Proof (lines 1-24)

(From a different run, so there may be minor differences.)

1:	$y=0\land i=0\land n\geq 0$	assumption
2:	$y=0$	$\land$ elim 1
3:	$i=0$	$\land$ elim 1
4:	$n\geq 0$	$\land$ elim 1
5:	$y=i\times i$	obviously, from 3,2
6:	$i\leq n$	obviously, from 4,3
7:	$i\geq 0$	obviously, from 3
8:	$1=2\times i+1$	obviously, from 3
9:	$y=i\times i\land i\leq n\land i\geq 0\land 1=2\times i+1$	$\land$ intro 5,6,7,8
10:	$y=0\land i=0\land n\geq 0\rightarrow y=i\times i\land i\leq n\land i\geq 0\land 1=2\times i+1$	$\rightarrow$ intro 1-9
11:	$\{y=i\times i\land i\leq n\land i\geq 0\land 1=2\times i+1\}(j:=1)\{y=i\times i\land i\leq n\land i\geq 0\land j=2\times i+1\}$	variable-assignment

Initialization

12:	$\{y=0\land i=0\land n\geq 0\}(j:=1)\{y=i\times i\land i\leq n\land i\geq 0\land j=2\times i+1\}$	consequence(L) 10,11
13:	$y=i\times i\land i\leq n\land i\geq 0\land j=2\times i+1\land i<n$	assumption
14:	$y=i\times i$	$\land$ elim 13
15:	$i\geq 0$	$\land$ elim 13
16:	$j=2\times i+1$	$\land$ elim 13
17:	$i<n$	$\land$ elim 13
18:	$y+j=(i+1)\times(i+1)$	obviously, from 16,14
19:	$i+1\leq n$	obviously, from 17
20:	$i+1\geq 0$	obviously, from 15
21:	$j+2=2\times(i+1)+1$	obviously, from 16
22:	$y+j=(i+1)\times(i+1)\land i+1\leq n\land i+1\geq 0\land j+2=2\times(i+1)+1$	$\land$ intro 18,19,20,21
23:	$y=i\times i\land i\leq n\land i\geq 0\land j=2\times i+1\land i<n\rightarrow y+j=(i+1)\times(i+1)\land i+1\leq n\land i+1\geq 0\land j+2=2\times(i+1)+1$	$\rightarrow$ intro 13-22
24:	$\{y+j=(i+1)\times(i+1)\land i+1\leq n\land i+1\geq 0\land j+2=2\times(i+1)+1\}(y:=y+j)\{y=(i+1)\times(i+1)\land i+1\leq n\land i+1\geq 0\land j+2=2\times(i+1)+1\}$	variable-assignment

First assignment in loop body

The Completed Proof (lines 25-51)

25:	$\{y=i\times i\land i\le n\land i\ge 0\land j=2\times i+1\land i<n\}(y:=y+j)\{y=(i+1)\times(i+1)\land i+1\le n\land i+1\ge 0\land j+2=2\times(i+1)+1\}$	consequence(L) 23,24	Other assignments in loop body
26:	$\{y=(i+1)\times(i+1)\land i+1\le n\land i+1\ge 0\land j+2=2\times(i+1)+1\}(j:=j+2)\{y=(i+1)\times(i+1)\land i+1\le n\land i+1\ge 0\land j=2\times(i+1)+1\}$	variable-assignment	
27:	$\{y=(i+1)\times(i+1)\land i+1\le n\land i+1\ge 0\land j=2\times(i+1)+1\}(i:=i+1)\{y=i\times i\land i\le n\land i\ge 0\land j=2\times i+1\}$	variable-assignment	
28:	$\{y=i\times i\land i\le n\land i\ge 0\land j=2\times i+1\land i<n\}(y:=y+j;j:=j+2;i:=i+1)\{y=i\times i\land i\le n\land i\ge 0\land j=2\times i+1\}$	sequence 25,26,27	
29:	$y=i\times i\land i\le n\land i\ge 0\land j=2\times i+1\land i<n$	assumption	Continuation condition on _M
30:	$i<n$	$\land$ elim 29	
31:	$n-i>0$	obviously, from 30	
32:	$y=i\times i\land i\le n\land i\ge 0\land j=2\times i+1\land i<n\rightarrow n-i>0$	$\rightarrow$ intro 29-31	
33:	integer Km	assumption	Termination of loop body
34:	$y=i\times i\land i\le n\land i\ge 0\land j=2\times i+1\land i<n\land n-i=Km$	assumption	
35:	$n-i=Km$	$\land$ elim 34	
36:	$n-(i+1)<Km$	obviously, from 35	
37:	$y=i\times i\land i\le n\land i\ge 0\land j=2\times i+1\land i<n\land n-i=Km\rightarrow n-(i+1)<Km$	$\rightarrow$ intro 34-36	
38:	$\{n-(i+1)<Km\}(y:=y+j)\{n-(i+1)<Km\}$	variable-assignment	
39:	$\{y=i\times i\land i\le n\land i\ge 0\land j=2\times i+1\land i<n\land n-i=Km\}(y:=y+j)\{n-(i+1)<Km\}$	consequence(L) 37,38	
40:	$\{n-(i+1)<Km\}(j:=j+2)\{n-(i+1)<Km\}$	variable-assignment	
41:	$\{n-(i+1)<Km\}(i:=i+1)\{n-i<Km\}$	variable-assignment	
42:	$\{y=i\times i\land i\le n\land i\ge 0\land j=2\times i+1\land i<n\land n-i=Km\}(y:=y+j;j:=j+2;i:=i+1)\{n-i<Km\}$	sequence 39,40,41	
43:	$\{y=i\times i\land i\le n\land i\ge 0\land j=2\times i+1\}\text{while } i<n \text{ do } y:=y+j;j:=j+2;i:=i+1 \text{ od}\{y=i\times i\land i\le n\land i\ge 0\land j=2\times i+1\land\neg(i<n)\}$	while 28,32,33-42	
44:	$y=i\times i\land i\le n\land i\ge 0\land j=2\times i+1\land\neg(i<n)$	assumption	Exit consequence
45:	$y=i\times i$	$\land$ elim 44	
46:	$i\le n$	$\land$ elim 44	
47:	$\neg(i<n)$	$\land$ elim 44	
48:	$y=n\times n$	obviously, from 47,46,45	
49:	$y=i\times i\land i\le n\land i\ge 0\land j=2\times i+1\land\neg(i<n)\rightarrow y=n\times n$	$\rightarrow$ intro 44-48	
50:	$\{y=i\times i\land i\le n\land i\ge 0\land j=2\times i+1\}\text{while } i<n \text{ do } y:=y+j;j:=j+2;i:=i+1 \text{ od}\{y=n\times n\}$	consequence(R) 43,49	
51:	$\{y=0\land i=0\land n\ge 0\}(j:=1)\{y=i\times i\land i\le n\land i\ge 0\land j=2\times i+1\}\text{while } i<n \text{ do } y:=y+j;j:=j+2;i:=i+1 \text{ od}\{y=n\times n\}$	Ntuple 12,50	

Verifying Array Programs

- Arrays present extra challenges and interesting issues.

- A useful dichotomy:
  - Programs with read-only arrays
  - Programs with modifiable arrays

Array Mathematics

- An array can be treated as if a **function**:
  - It maps indices into values.
  - e.g. a 1-dimensional array with dimension 10 maps {0, …, 9} into values of the type stored in the array.

  - a[i] is the value of this function with argument I

Read-Only Array Example: Finding a zero element

$\{n \geq 0 \wedge \text{length}(a)=n\}$

```
i := 0; j := n
```

$\{i \leq n \wedge i \geq 0 \wedge \text{length}(a)=n \wedge j<n \rightarrow a[j] = 0\}$

```
while i < n do
        if a[i] = 0
                then j := i
                else skip fi;
        i := i+1 od
```

$\{j<n \rightarrow a[j] = 0\}$

JAPE Type-in (for reference)

```
WHERE DISTINCT i,j,n,a IS
{0≤n∧length(a)=n}
(i := 0; j := n)
{i≤n∧0≤i∧length(a)=n∧(j<n→a[j]=0)}
while i < n
do
if a[i] = 0 then j := i else skip fi;
i := i+1
od
{j < n →a[j] = 0}
```

Salient Parts of Proof Initialization

1: $0 \le n \wedge \mathrm{length}(a)=n$

2: $0 \le n$

3: $\mathrm{length}(a)=n$

4: $0 \le 0$

5: $n<n$

6: $\perp$

7: $a[n]=0$

8: $n<n \rightarrow a[n]=0$

9: $0 \le n \wedge 0 \le 0 \wedge \mathrm{length}(a)=n \wedge (n<n \rightarrow a[n]=0)$

10: $0 \le n \wedge \mathrm{length}(a)=n \rightarrow 0 \le n \wedge 0 \le 0 \wedge \mathrm{length}(a)=n \wedge (n<n \rightarrow a[n]=0)$

11: $\{0 \le n \wedge 0 \le 0 \wedge \mathrm{length}(a)=n \wedge (n<n \rightarrow a[n]=0)\}(i:=0)\{i \le n \wedge 0 \le i \wedge \mathrm{length}(a)=n \wedge (n<n \rightarrow a[n]=0)\}$

12: $\{0 \le n \wedge \mathrm{length}(a)=n\}(i:=0)\{i \le n \wedge 0 \le i \wedge \mathrm{length}(a)=n \wedge (n<n \rightarrow a[n]=0)\}$

13: $\{i \le n \wedge 0 \le i \wedge \mathrm{length}(a)=n \wedge (n<n \rightarrow a[n]=0)\}(j:=n)\{i \le n \wedge 0 \le i \wedge \mathrm{length}(a)=n \wedge (j<n \rightarrow a[j]=0)\}$

14: $\{0 \le n \wedge \mathrm{length}(a)=n\}(i:=0;j:=n)\{i \le n \wedge 0 \le i \wedge \mathrm{length}(a)=n \wedge (j<n \rightarrow a[j]=0)\}$

Loop Body Partial Correctness

(Collapsed)

15:	$i \leq n \wedge 0 \leq i \wedge length(a)=n \wedge (j<n \rightarrow a[j]=0) \wedge i<n$	assumption
16:	$(a[i]=0 \rightarrow i+1 \leq n \wedge 0 \leq i+1 \wedge length(a)=n \wedge (i<n \rightarrow a[i]=0)) \wedge (\neg(a[i]=0) \rightarrow i+1 \leq n \wedge 0 \leq i+1 \wedge length(a)=n \wedge (j<n \rightarrow a[j]=0)) \wedge 0 \leq i \wedge i<length(a)$	{cut}
17:	$i \leq n \wedge 0 \leq i \wedge length(a)=n \wedge (j<n \rightarrow a[j]=0) \wedge i<n \rightarrow (a[i]=0 \rightarrow i+1 \leq n \wedge 0 \leq i+1 \wedge length(a)=n \wedge (i<n \rightarrow a[i]=0)) \wedge (\neg(a[i]=0) \rightarrow i+1 \leq n \wedge 0 \leq i+1 \wedge length(a)=n \wedge (j<n \rightarrow a[j]=0)) \wedge 0 \leq i \wedge i<length(a)$	→ intro 15-16
18:	$\{i+1 \leq n \wedge 0 \leq i+1 \wedge length(a)=n \wedge (i<n \rightarrow a[i]=0)\}(j:=i)\{i+1 \leq n \wedge 0 \leq i+1 \wedge length(a)=n \wedge (j<n \rightarrow a[j]=0)\}$	variable-assignment
19:	$\{i+1 \leq n \wedge 0 \leq i+1 \wedge length(a)=n \wedge (j<n \rightarrow a[j]=0)\}skip\{i+1 \leq n \wedge 0 \leq i+1 \wedge length(a)=n \wedge (j<n \rightarrow a[j]=0)\}$	skip
20:	$\{(a[i]=0 \rightarrow i+1 \leq n \wedge 0 \leq i+1 \wedge length(a)=n \wedge (i<n \rightarrow a[i]=0)) \wedge (\neg(a[i]=0) \rightarrow i+1 \leq n \wedge 0 \leq i+1 \wedge length(a)=n \wedge (j<n \rightarrow a[j]=0)) \wedge 0 \leq i \wedge i<length(a)\}\text{if } a[i]=0 \text{ then } j:=i \text{ else skip fi}\{i+1 \leq n \wedge 0 \leq i+1 \wedge length(a)=n \wedge (j<n \rightarrow a[j]=0)\}$	choice 18,19
21:	$\{i \leq n \wedge 0 \leq i \wedge length(a)=n \wedge (j<n \rightarrow a[j]=0) \wedge i<n\}\text{if } a[i]=0 \text{ then } j:=i \text{ else skip fi}\{i+1 \leq n \wedge 0 \leq i+1 \wedge length(a)=n \wedge (j<n \rightarrow a[j]=0)\}$	consequence(L) 17,20
22:	$\{i+1 \leq n \wedge 0 \leq i+1 \wedge length(a)=n \wedge (j<n \rightarrow a[j]=0)\}(i:=i+1)\{i \leq n \wedge 0 \leq i \wedge length(a)=n \wedge (j<n \rightarrow a[j]=0)\}$	variable-assignment
23	$\boxed{\{i \leq n \wedge 0 \leq i \wedge length(a)=n \wedge (j<n \rightarrow a[j]=0) \wedge i<n\}(\text{if } a[i]=0 \text{ then } j:=i \text{ else skip fi};i:=i+1)\{i \leq n \wedge 0 \leq i \wedge length(a)=n \wedge (j<n \rightarrow a[j]=0)\}}$	sequence 21,22

Variant Continuation and Decrease

24:	$i\le n \wedge 0\le i \wedge \mathrm{length}(a)=n \wedge (j<n \to a[j]=0) \wedge i<n$	assumption
25:	$n-i>0$	{cut}
26:	$i\le n \wedge 0\le i \wedge \mathrm{length}(a)=n \wedge (j<n \to a[j]=0) \wedge i<n \to n-i>0$	$\to$ intro 24–25

(Collapsed)

27:	integer Km	assumption
28:	$\{i\le n \wedge 0\le i \wedge \mathrm{length}(a)=n \wedge (j<n \to a[j]=0) \wedge i<n \wedge n-i=Km\}(\text{if } a[i]=0 \text{ then } j:=i \text{ else skip fi}; i:=i+1)\{n-i<Km\}$	{cut}

Remainder

29:	$\{i\le n \wedge 0\le i \wedge \mathrm{length}(a)=n \wedge (j<n \to a[j]=0)\}\text{while } i<n \text{ do if } a[i]=0 \text{ then } j:=i \text{ else skip fi}; i:=i+1 \text{ od}$ $\{i\le n \wedge 0\le i \wedge \mathrm{length}(a)=n \wedge (j<n \to a[j]=0) \wedge \neg(i<n)\}$	while 23,26,27–28
30:	$i\le n \wedge 0\le i \wedge \mathrm{length}(a)=n \wedge (j<n \to a[j]=0) \wedge \neg(i<n)$	assumption
31:	$j<n \to a[j]=0$	$\wedge$ elim 30
32:	$i\le n \wedge 0\le i \wedge \mathrm{length}(a)=n \wedge (j<n \to a[j]=0) \wedge \neg(i<n) \to j<n \to a[j]=0$	$\to$ intro 30–31
33:	$\{i\le n \wedge 0\le i \wedge \mathrm{length}(a)=n \wedge (j<n \to a[j]=0)\}\text{while } i<n \text{ do if } a[i]=0 \text{ then } j:=i \text{ else skip fi}; i:=i+1 \text{ od}\{j<n \to a[j]=0\}$	consequence(R) 29,32
34:	$\{0\le n \wedge \mathrm{length}(a)=n\}(i:=0; j:=n)\{i\le n \wedge 0\le i \wedge \mathrm{length}(a)=n \wedge (j<n \to a[j]=0)\}$ $\text{while } i<n \text{ do if } a[i]=0 \text{ then } j:=i \text{ else skip fi}; i:=i+1 \text{ od}\{j<n \to a[j]=0\}$	Ntuple 14,33

Without collapsing, the proof is about 74 lines.

Quantifiers

- Quantifiers are handy representing information about arrays, e.g.

- $\forall i \; ((0 < i) \wedge (i < n)) \rightarrow a[i-1] \leq a[i]$

- $\exists i \; ((0 \leq i) \wedge (i < n) \wedge a[i] = 0)$

Read-Only Example with Quantifiers

A program whose correct working depends on values stored in the array.

$\{\boxed{\exists x.(0 \leq x \wedge x < \mathrm{length}(a) \wedge a[x]=0)}\}(i:=0)$

$\{0 \leq i \wedge i < \mathrm{length}(a) \wedge \exists x.(i \leq x \wedge x < \mathrm{length}(a) \wedge a[x]=0)\}$while $a[i] \neq 0$ do $i:=i+1$ od$\{a[i]=0\}$

- The **assumption** is that the array contains a **zero element**.
- The **expectation** is to find the index of such an element.
- **Termination critically depends on this assumption**.
- The **invariant** says that the element such that **a[x] = 0 is still to be found**.

Start of Proof

...

$\{\exists x.(0 \leq x \wedge x < length(a) \wedge a[x]=0)\}(i:=0)$

1: $\{0 \leq i \wedge i < length(a) \wedge \exists x.(i \leq x \wedge x < length(a) \wedge a[x]=0)\}$

while $a[i] \neq 0$ do $i:=i+1$ od$\{a[i]=0\}$

Consequence for Initialization

1:	$\exists x.(0 \le x \wedge x < length(a) \wedge a[x]=0)$	assumption
	...	
2:	$0 \le 0 \wedge 0 < length(a) \wedge \exists x.(0 \le x \wedge x < length(a) \wedge a[x]=0)$	

We'll use ∃ elim.

1:	$\exists x.(0 \le x \wedge x < length(a) \wedge a[x]=0)$	assumption
2:	integer i1	assumption
3:	$0 \le i1 \wedge i1 < length(a) \wedge a[i1]=0$	assumption
	...	
4:	$0 \le 0 \wedge 0 < length(a) \wedge \exists x.(0 \le x \wedge x < length(a) \wedge a[x]=0)$	
5:	$0 \le 0 \wedge 0 < length(a) \wedge \exists x.(0 \le x \wedge x < length(a) \wedge a[x]=0)$	∃ elim 1,2-4

Done with Initialization

1:	$\exists x.(0 \le x \wedge x < length(a) \wedge a[x]=0)$	assumption
2:	integer i1	assumption
3:	$0 \le i1 \wedge i1 < length(a) \wedge a[i1]=0$	assumption
4:	$0 \le i1$	$\wedge$ elim 3
5:	$i1 < length(a)$	$\wedge$ elim 3
6:	$0 \le 0$	$A \le A$
7:	$0 < length(a)$	$A \le B, B < C \vdash A < C$ 4,5
8:	$0 \le 0 \wedge 0 < length(a) \wedge \exists x.(0 \le x \wedge x < length(a) \wedge a[x]=0)$	$\wedge$ intro 6,7,1
9:	$0 \le 0 \wedge 0 < length(a) \wedge \exists x.(0 \le x \wedge x < length(a) \wedge a[x]=0)$	$\exists$ elim 1,2-8
10:	$\exists x.(0 \le x \wedge x < length(a) \wedge a[x]=0)$ $\rightarrow 0 \le 0 \wedge 0 < length(a) \wedge \exists x.(0 \le x \wedge x < length(a) \wedge a[x]=0)$	$\rightarrow$ intro 1-9
11:	$\{0 \le 0 \wedge 0 < length(a) \wedge \exists x.(0 \le x \wedge x < length(a) \wedge a[x]=0)\}$ $(i:=0)\{0 \le i \wedge i < length(a) \wedge \exists x.(i \le x \wedge x < length(a) \wedge a[x]=0)\}$	variable-assignment
12:	$\{\exists x.(0 \le x \wedge x < length(a) \wedge a[x]=0)\}(i:=0)$ $\{0 \le i \wedge i < length(a) \wedge \exists x.(i \le x \wedge x < length(a) \wedge a[x]=0)\}$	consequence(L) 10,11

On to the while loop

13: $\{0 \leq i \wedge i < \text{length}(a) \wedge \exists x.(i \leq x \wedge x < \text{length}(a) \wedge a[x]=0)\}$
while $a[i] \neq 0$ do $i:=i+1$ od$\{a[i]=0\}$

A subtlety arises in proving this invariant.

If the loop does not stop, it means $a[i] \neq 0$.

This is used to re-establish the invariant.

Body Partial Correctness

...

14: $\{0\leq i \wedge i<\text{length}(a) \wedge \exists x.(i\leq x \wedge x<\text{length}(a) \wedge a[x]=0) \wedge a[i]\neq 0\}$
$(i:=i+1)\{0\leq i \wedge i<\text{length}(a) \wedge \exists x.(i\leq x \wedge x<\text{length}(a) \wedge a[x]=0)\}$

Use the assignment rule.
This generates a consequent implication.

16: $0\leq i \wedge i<\text{length}(a) \wedge \exists x.(i\leq x \wedge x<\text{length}(a) \wedge a[x]=0) \wedge a[i]\neq 0$ assumption

...

17: $0\leq i+1 \wedge i+1<\text{length}(a) \wedge \exists x.(i+1\leq x \wedge x<\text{length}(a) \wedge a[x]=0)$

18: $0\leq i \wedge i<\text{length}(a) \wedge \exists x.(i\leq x \wedge x<\text{length}(a) \wedge a[x]=0) \wedge a[i]\neq 0$
$\rightarrow 0\leq i+1 \wedge i+1<\text{length}(a) \wedge \exists x.(i+1\leq x \wedge x<\text{length}(a) \wedge a[x]=0)$ $\rightarrow$ intro 16–17

It is tempting to try backward $\wedge$ Introduction now. Don't!

Use Forward $\wedge$ Elimination

16:	$0 \le i \wedge i < \mathrm{length}(a) \wedge \exists x.(i \le x \wedge x < \mathrm{length}(a) \wedge a[x] = 0) \wedge a[i] \ne 0$	assumption
17:	$0 \le i$	$\wedge$ elim 16
18:	$i < \mathrm{length}(a)$	$\wedge$ elim 16
19:	$\exists x.(i \le x \wedge x < \mathrm{length}(a) \wedge a[x] = 0)$	$\wedge$ elim 16
20:	$a[i] \ne 0$	$\wedge$ elim 16
	...	
21:	$0 \le i+1 \wedge i+1 < \mathrm{length}(a) \wedge \exists x.(i+1 \le x \wedge x < \mathrm{length}(a) \wedge a[x] = 0)$	

Now use ∃ Elimination, introducing i2

16:	$0 \le i \wedge i < length(a) \wedge \exists x.(i \le x \wedge x < length(a) \wedge a[x]=0) \wedge a[i] \ne 0$	assumption
17:	$0 \le i$	$\wedge$ elim 16
18:	$i < length(a)$	$\wedge$ elim 16
19:	$\exists x.(i \le x \wedge x < length(a) \wedge a[x]=0)$	$\wedge$ elim 16
20:	$a[i] \ne 0$	$\wedge$ elim 16
21:	integer i2	assumption
22:	$i \le i2 \wedge i2 < length(a) \wedge a[i2]=0$	assumption
	...	
23:	$0 \le i+1 \wedge i+1 < length(a) \wedge \exists x.(i+1 \le x \wedge x < length(a) \wedge a[x]=0)$	
24:	$0 \le i+1 \wedge i+1 < length(a) \wedge \exists x.(i+1 \le x \wedge x < length(a) \wedge a[x]=0)$	$\exists$ elim 19,21-23

Note that i2 cannot equal i. Why?
We need to show this, to get the post-condition.

To get i2≠i

- Set up a bifurcation using i ≤ i2, i.e. i < i2 ∨ i = i2.
- Then use ∨ Elimination:

20:	$a[i]\neq 0$	∧ elim 16
21:	integer i2	assumption
22:	$i\leq i2 \wedge i2<length(a) \wedge a[i2]=0$	assumption
23:	$i\leq i2$	∧ elim 22
24:	$i<i2 \vee i=i2$	$A\leq B \triangleq A<B \vee A=B$ 23
25:	$i2<length(a)$	∧ elim 22
26:	$a[i2]=0$	∧ elim 22
27:	$i<i2$	assumption
	...	
28:	$0\leq i+1 \wedge i+1<length(a) \wedge \exists x.(i+1\leq x \wedge x<length(a) \wedge a[x]=0)$	
29:	$i=i2$	assumption
	...	
30:	$0\leq i+1 \wedge i+1<length(a) \wedge \exists x.(i+1\leq x \wedge x<length(a) \wedge a[x]=0)$	
31:	$0\leq i+1 \wedge i+1<length(a) \wedge \exists x.(i+1\leq x \wedge x<length(a) \wedge a[x]=0)$	∨ elim 24,27-28,29-30
32:	$0\leq i+1 \wedge i+1<length(a) \wedge \exists x.(i+1\leq x \wedge x<length(a) \wedge a[x]=0)$	∃ elim 19,21-31

Now try backward ∧ Introductions.

- What to use for x in ∃x…?

26:	$a[i2]=0$	∧ elim 22
27:	$i<i2$	assumption
	...	
28:	$0\leq i+1$	
	...	
29:	$i+1<length(a)$	
	...	
30:	$\exists x.(i+1\leq x \wedge x<length(a) \wedge a[x]=0)$	
31:	$0\leq i+1 \wedge i+1<length(a) \wedge \exists x.(i+1\leq x \wedge x<length(a) \wedge a[x]=0)$	∧ intro 28,29,30

i2 is the obvious choice for x

23:	$i \leq i2$	$\wedge$ elim 22
24:	$i < i2 \vee i = i2$	$A \leq B \triangleq A < B \vee A = B$ 23
25:	$i2 < length(a)$	$\wedge$ elim 22
26:	$a[i2] = 0$	$\wedge$ elim 22
27:	$i < i2$	assumption
	...	
28:	$0 \leq i+1$	
	...	
29:	$i+1 < length(a)$	
	...	
30:	$i+1 \leq i2 \wedge i2 < length(a) \wedge a[i2] = 0$	
31:	$\exists x.(i+1 \leq x \wedge x < length(a) \wedge a[x] = 0)$	$\exists$ intro 30
32:	$0 \leq i+1 \wedge i+1 < length(a) \wedge \exists x.(i+1 \leq x \wedge x < length(a) \wedge a[x] = 0)$	$\wedge$ intro 28,29,31

Closure of the top half

23:	$i \le i2$	$\wedge$ elim 22
24:	$i < i2 \vee i = i2$	$A \le B \triangleq A < B \vee A = B$ 23
25:	$i2 < length(a)$	$\wedge$ elim 22
26:	$a[i2] = 0$	$\wedge$ elim 22
27:	$i < i2$	assumption
28:	$0 \le i+1$	$A \le B \vdash A \le B+1$ 17
29:	$i+1 < length(a)$	$i < i2,\ i2 < length(a) \vdash \ldots$ 27,25
30:	$i+1 \le i2$	$A < N \vdash A+1 \le N$ 27
31:	$i+1 \le i2 \wedge i2 < length(a) \wedge a[i2] = 0$	$\wedge$ intro 30,25,26
32:	$\exists x.(i+1 \le x \wedge x < length(a) \wedge a[x] = 0)$	$\exists$ intro 31
33:	$0 \le i+1 \wedge i+1 < length(a) \wedge \exists x.(i+1 \le x \wedge x < length(a) \wedge a[x] = 0)$	$\wedge$ intro 28,29,32

Bottom Half: We want a contradiction

20:	$a[i]\neq 0$	$\wedge$ elim 16
21:	integer i2	assumption
22:	$i\leq i2 \wedge i2<length(a) \wedge a[i2]=0$	assumption
23:	$i\leq i2$	$\wedge$ elim 22
24:	$i<i2 \vee i=i2$	$A\leq B \triangleq A<B \vee A=B$ 23
25:	$i2<length(a)$	$\wedge$ elim 22
26:	$a[i2]=0$	$\wedge$ elim 22

34:	$i=i2$
	...
35:	$0\leq i+1 \wedge i+1<length(a) \wedge \exists x.(i+1\leq x \wedge x<length(a) \wedge a[x]=0)$

Introduce $\perp$, then figure out how to get it.

How to pick _B1?

20:	$a[i]\neq 0$	
21:	integer i2	
22:	$i\leq i2 \wedge i2<length(a) \wedge a[i2]=0$	
23:	$i\leq i2$	
24:	$i<i2 \vee i=i2$	
25:	$i2<length(a)$	
26:	$a[i2]=0$	

34:	$i=i2$	assumption
	...	
35:	_B1	
	...	
36:	¬_B1	
37:	$\perp$	¬ elim 35,36
38:	$0\leq i+1 \wedge i+1<length(a) \wedge \exists x.(i+1\leq x \wedge x<length(a) \wedge a[x]=0)$	contra (constructive) 37

For _B1 use a[i] = 0

34:	$i = i2$
	...
35:	$a[i] = 0$
	...
36:	$\neg(a[i] = 0)$
37:	$\bot$
38:	$0 \le i+1 \wedge i+1 < length(a) \wedge \exists x.(i+1 \le x \wedge x < length(a) \wedge a[x] = 0)$

Then use a comparison rule to get the second conclusion from a[i] $\neq$ 0:

$$A \neq B \triangleq \neg(A = B)$$

Status

26:	$a[i2]=0$	$\wedge$ elim 22
34:	$i=i2$	assumption
	...	
35:	$a[i]=0$	
36:	$\neg(a[i]=0)$	$A \neq B \triangleq \neg(A=B)$ 20
37:	$\perp$	$\neg$ elim 35,36
38:	$0 \leq i+1 \wedge i+1 < length(a) \wedge \exists x.(i+1 \leq x \wedge x < length(a) \wedge a[x]=0)$	contra (constructive) 37

Now use equality, to get 35 from 26 and 34.

Closure

26:	$a[i2]=0$	$\wedge$ elim 22
33:	$i=i2$	assumption
34:	$a[i]=0$	equality-substitution 33,25
35:	$\neg(a[i]=0)$	$A \neq B \triangleq \neg(A=B)$ 19
36:	$\perp$	$\neg$ elim 34,35
37:	$0 \leq i+1 \wedge i+1 < length(a) \wedge \exists x.(i+1 \leq x \wedge x < length(a) \wedge a[x]=0)$	contra (constructive) 36

What about termination?

- What to use for _M?

Completed proof, using several lemmas 1/3

1:	$\exists x.(0\leq x \wedge x<\mathrm{length}(a) \wedge a[x]=0)$	assumption
2:	integer i1	assumption
3:	$0\leq i1 \wedge i1<\mathrm{length}(a) \wedge a[i1]=0$	assumption
4:	$0\leq i1$	$\wedge$ elim 3
5:	$i1<\mathrm{length}(a)$	$\wedge$ elim 3
6:	$0\leq 0$	$A\leq A$
7:	$0<\mathrm{length}(a)$	$A\leq B, B<C \vdash A<C$ 4,5
8:	$0\leq 0 \wedge 0<\mathrm{length}(a) \wedge \exists x.(0\leq x \wedge x<\mathrm{length}(a) \wedge a[x]=0)$	$\wedge$ intro 6,7,1
9:	$0\leq 0 \wedge 0<\mathrm{length}(a) \wedge \exists x.(0\leq x \wedge x<\mathrm{length}(a) \wedge a[x]=0)$	$\exists$ elim 1,2-8
10:	$\exists x.(0\leq x \wedge x<\mathrm{length}(a) \wedge a[x]=0) \rightarrow 0\leq 0 \wedge 0<\mathrm{length}(a) \wedge \exists x.(0\leq x \wedge x<\mathrm{length}(a) \wedge a[x]=0)$	$\rightarrow$ intro 1-9
11:	$\{0\leq 0 \wedge 0<\mathrm{length}(a) \wedge \exists x.(0\leq x \wedge x<\mathrm{length}(a) \wedge a[x]=0)\}$ $(i:=0)\{0\leq i \wedge i<\mathrm{length}(a) \wedge \exists x.(i\leq x \wedge x<\mathrm{length}(a) \wedge a[x]=0)\}$	variable-assignment
12:	$\{\exists x.(0\leq x \wedge x<\mathrm{length}(a) \wedge a[x]=0)\}(i:=0)\{0\leq i \wedge i<\mathrm{length}(a) \wedge \exists x.(i\leq x \wedge x<\mathrm{length}(a) \wedge a[x]=0)\}$	consequence(L) 10,11
13:	$0\leq i \wedge i<\mathrm{length}(a) \wedge \exists x.(i\leq x \wedge x<\mathrm{length}(a) \wedge a[x]=0)$	assumption
14:	$0\leq i \wedge i<\mathrm{length}(a)$	$\wedge$ elim(L) 13
15:	$0\leq i \wedge i<\mathrm{length}(a) \wedge \exists x.(i\leq x \wedge x<\mathrm{length}(a) \wedge a[x]=0) \rightarrow 0\leq i \wedge i<\mathrm{length}(a)$	$\rightarrow$ intro 13-14

16:	$0 \le i \land i < \mathrm{length}(a) \land \exists x.(i \le x \land x < \mathrm{length}(a) \land a[x]=0) \land a[i] \ne 0$	assumption
17:	$0 \le i$	$\land$ elim 16
18:	$\exists x.(i \le x \land x < \mathrm{length}(a) \land a[x]=0)$	$\land$ elim 16
19:	$a[i] \ne 0$	$\land$ elim 16
20:	integer i2	assumption
21:	$i \le i2 \land i2 < \mathrm{length}(a) \land a[i2]=0$	assumption
22:	$i \le i2$	$\land$ elim 21
23:	$i < i2 \lor i = i2$	$A \le B \triangleq A < B \lor A = B$ 22
24:	$i2 < \mathrm{length}(a)$	$\land$ elim 21
25:	$a[i2]=0$	$\land$ elim 21
26:	$i < i2$	assumption
27:	$0 \le i+1$	$A \le B \vdash A \le B+1$ 17
28:	$i+1 < \mathrm{length}(a)$	$i<i2,\ i2<\mathrm{length}(a)$ ... 26,24
29:	$i+1 \le i2$	$A<N \vdash A+1 \le N$ 26
30:	$i+1 \le i2 \land i2 < \mathrm{length}(a) \land a[i2]=0$	$\land$ intro 29,24,25
31:	$\exists x.(i+1 \le x \land x < \mathrm{length}(a) \land a[x]=0)$	$\exists$ intro 30
32:	$0 \le i+1 \land i+1 < \mathrm{length}(a) \land \exists x.(i+1 \le x \land x < \mathrm{length}(a) \land a[x]=0)$	$\land$ intro 27,28,31
33:	$i = i2$	assumption
34:	$a[i]=0$	equality-substitution 33,25
35:	$\neg(a[i]=0)$	$A \ne B \triangleq \neg(A=B)$ 19
36:	$\bot$	$\neg$ elim 34,35
37:	$0 \le i+1 \land i+1 < \mathrm{length}(a) \land \exists x.(i+1 \le x \land x < \mathrm{length}(a) \land a[x]=0)$	contra (constructive) 36
38:	$0 \le i+1 \land i+1 < \mathrm{length}(a) \land \exists x.(i+1 \le x \land x < \mathrm{length}(a) \land a[x]=0)$	$\lor$ elim 23,26-32,33-37
39:	$0 \le i+1 \land i+1 < \mathrm{length}(a) \land \exists x.(i+1 \le x \land x < \mathrm{length}(a) \land a[x]=0)$	$\exists$ elim 18,20-38

40:	$0\le i\wedge i<\mathrm{length}(a)\wedge\exists x.(i\le x\wedge x<\mathrm{length}(a)\wedge a[x]=0)\wedge a[i]\ne 0$ $\rightarrow 0\le i+1\wedge i+1<\mathrm{length}(a)\wedge\exists x.(i+1\le x\wedge x<\mathrm{length}(a)\wedge a[x]=0)$	$\rightarrow$ intro 16-39
41:	$\{0\le i+1\wedge i+1<\mathrm{length}(a)\wedge\exists x.(i+1\le x\wedge x<\mathrm{length}(a)\wedge a[x]=0)\}$ $(i:=i+1)\{0\le i\wedge i<\mathrm{length}(a)\wedge\exists x.(i\le x\wedge x<\mathrm{length}(a)\wedge a[x]=0)\}$	variable-assignment
42:	$\{0\le i\wedge i<\mathrm{length}(a)\wedge\exists x.(i\le x\wedge x<\mathrm{length}(a)\wedge a[x]=0)\wedge a[i]\ne 0\}$ $(i:=i+1)\{0\le i\wedge i<\mathrm{length}(a)\wedge\exists x.(i\le x\wedge x<\mathrm{length}(a)\wedge a[x]=0)\}$	consequence(L) 40,41
43:	$0\le i\wedge i<\mathrm{length}(a)\wedge\exists x.(i\le x\wedge x<\mathrm{length}(a)\wedge a[x]=0)\wedge a[i]\ne 0$	assumption
44:	$i<\mathrm{length}(a)$	$\wedge$ elim 43
45:	$\mathrm{length}(a)-i>0$	$i<\mathrm{length}(a)\vdash$ length(a... 44
46:	$0\le i\wedge i<\mathrm{length}(a)\wedge\exists x.(i\le x\wedge x<\mathrm{length}(a)\wedge a[x]=0)\wedge a[i]\ne 0\rightarrow\mathrm{length}(a)-i>0$	$\rightarrow$ intro 43-45
47:	integer Km	assumption
48:	$0\le i\wedge i<\mathrm{length}(a)\wedge\exists x.(i\le x\wedge x<\mathrm{length}(a)\wedge a[x]=0)\wedge a[i]\ne 0\wedge\mathrm{length}(a)-i=Km$	assumption
49:	$\mathrm{length}(a)-i=Km$	$\wedge$ elim 48
50:	$\mathrm{length}(a)-(i+1)<Km$	$\mathrm{length}(a)-i=X\vdash$ lengt... 49
51:	$0\le i\wedge i<\mathrm{length}(a)\wedge\exists x.(i\le x\wedge x<\mathrm{length}(a)\wedge a[x]=0)\wedge a[i]\ne 0\wedge\mathrm{length}(a)-i=Km$ $\rightarrow\mathrm{length}(a)-(i+1)<Km$	$\rightarrow$ intro 48-50
52:	$\{\mathrm{length}(a)-(i+1)<Km\}(i:=i+1)\{\mathrm{length}(a)-i<Km\}$	variable-assignment
53:	$\{0\le i\wedge i<\mathrm{length}(a)\wedge\exists x.(i\le x\wedge x<\mathrm{length}(a)\wedge a[x]=0)\wedge a[i]\ne 0\wedge\mathrm{length}(a)-i=Km\}$ $(i:=i+1)\{\mathrm{length}(a)-i<Km\}$	consequence(L) 51,52
54:	$\{0\le i\wedge i<\mathrm{length}(a)\wedge\exists x.(i\le x\wedge x<\mathrm{length}(a)\wedge a[x]=0)\}$ while $a[i]\ne 0$ do $i:=i+1$ od $\{0\le i\wedge i<\mathrm{length}(a)\wedge\exists x.(i\le x\wedge x<\mathrm{length}(a)\wedge a[x]=0)\wedge\neg(a[i]\ne 0)\}$	while 15,42,46,47-53
55:	$0\le i\wedge i<\mathrm{length}(a)\wedge\exists x.(i\le x\wedge x<\mathrm{length}(a)\wedge a[x]=0)\wedge\neg(a[i]\ne 0)$	assumption
56:	$\neg(a[i]\ne 0)$	$\wedge$ elim 55
57:	$a[i]=0$	$A=B\triangleq\neg(A\ne B)$ 56
58:	$0\le i\wedge i<\mathrm{length}(a)\wedge\exists x.(i\le x\wedge x<\mathrm{length}(a)\wedge a[x]=0)\wedge\neg(a[i]\ne 0)\rightarrow a[i]=0$	$\rightarrow$ intro 55-57
59:	$\{0\le i\wedge i<\mathrm{length}(a)\wedge\exists x.(i\le x\wedge x<\mathrm{length}(a)\wedge a[x]=0)\}$ while $a[i]\ne 0$ do $i:=i+1$ od $\{a[i]=0\}$	consequence(R) 54,58
60:	$\{\exists x.(0\le x\wedge x<\mathrm{length}(a)\wedge a[x]=0)\}(i:=0)$ $\{0\le i\wedge i<\mathrm{length}(a)\wedge\exists x.(i\le x\wedge x<\mathrm{length}(a)\wedge a[x]=0)\}$ while $a[i]\ne 0$ do $i:=i+1$ od $\{a[i]=0\}$	Ntuple 12,59

Modifiable Arrays

- Arrays are like functions.
- Assigning to an array element is like creating a new function.
- The new function differs from the old in that one element may be different from before.

JAPE Array Modification Notation

- Jape Notation: $\mathbf{a \oplus i \rightarrow v}$ is the array that is like **a** except that the value of **a[i]** is **v**.

- $(a \oplus i \rightarrow v)[i] = v$ same index, new value

- $(a \oplus i \rightarrow v)[j] = a[j]$ if $j \neq i$ different index, old value

Array Element Assignment

- Consider a triple

  {??}
  a[i] := E
  {B}

- If this is viewed mathematically as an array replacement:

  {??}
  a := $(a \oplus i \rightarrow E)$
  {B}

- then it is obvious what ?? should be.

Isn't it?

- If this is viewed mathematically as an array replacement:

  $\{??\}$
  $a := (a \oplus i \rightarrow E)$
  $\{B\}$

- then it is obvious what ?? should be:

  $B[(a \oplus i \rightarrow E)/a]$

- i.e. replace all free instances of a in B with $(a \oplus i \rightarrow E)/a$.

  $\{B[(a \oplus i \rightarrow E)/a]\}$
  $a := (a \oplus i \rightarrow E)$
  $\{B\}$

Example

- $\{??\}$
  $a[1] := 99$
  $\{a[0] = 98 \wedge a[1] > 50\}$

- $\{(a \oplus 1 \rightarrow 99)[0] = 98 \wedge (a \oplus 1 \rightarrow 99)[1] > 50\}$
  $a := (a \oplus 1 \rightarrow 99)$
  $\{a[0] = 98 \wedge a[1] > 50\}$

JAPE Example

...

1: $\{\_Unknown\}(a[1]:=99)\{a[0]=98 \wedge a[1]>50\}$

- $\{??\}$
  $a[1] := 99$
  $\{a[0] = 98 \wedge a[1] > 50\}$

- $\{(a\oplus 1\rightarrow 99)[0] = 98 \wedge (a\oplus 1\rightarrow 99)[1] > 50\}$
  $a := (a\oplus 1\rightarrow 99)$
  $\{a[0] = 98 \wedge a[1] > 50\}$

1: $\{(a\oplus 1\mapsto 99)[0]=98 \wedge (a\oplus 1\mapsto 99)[1]>50 \wedge 0\leq 1 \wedge 1<length(a)\}$ array-element-assignment
$(a[1]:=99)\{a[0]=98 \wedge a[1]>50\}$

Bornat's Array Assignment Rule in JAPE

assignment

$$\{\ (a[E]\ \mathit{computes}) \wedge (F\ \mathit{computes})\ \wedge B^{a}_{a \oplus E \mapsto F}\ \}\ a[E] := F\ \{B\}$$

Formula B with each
free a replaced with a⊕E→F

B

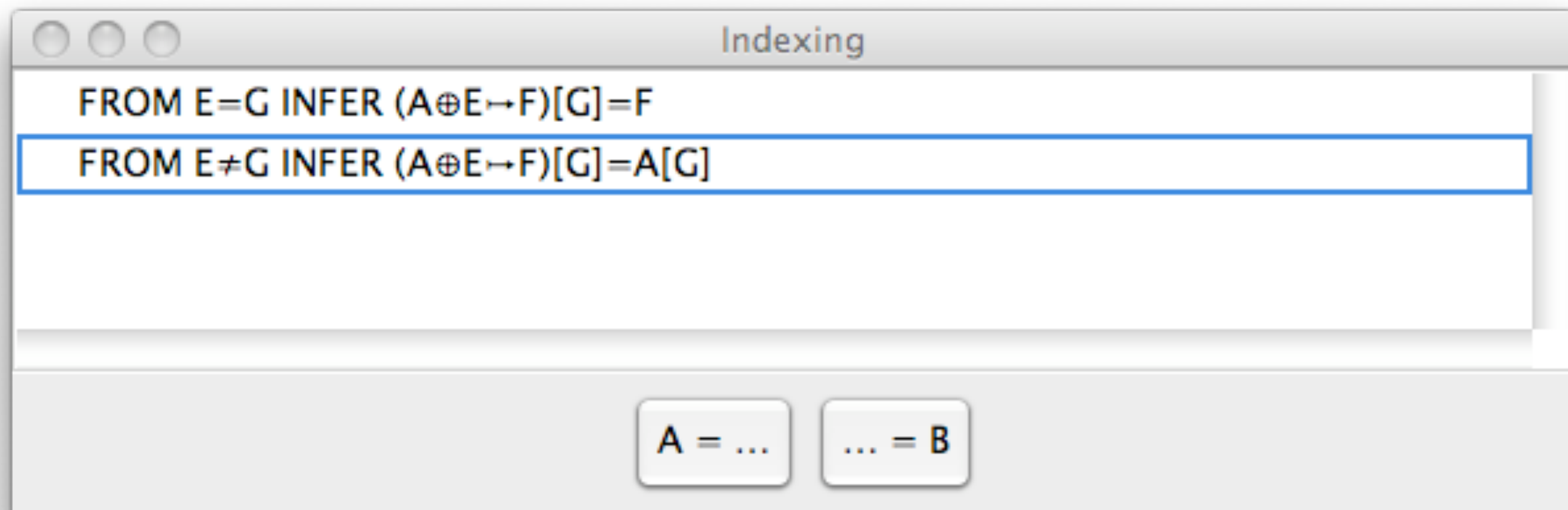
What *a[E] computes* means:

$$\frac{\begin{array}{c}\vdots \\ E\ \mathit{computes}\end{array} \qquad \begin{array}{c}\vdots \\ 0 \leq E < \mathrm{length}(a)\end{array}}{a[E]\ \mathit{computes}}$$

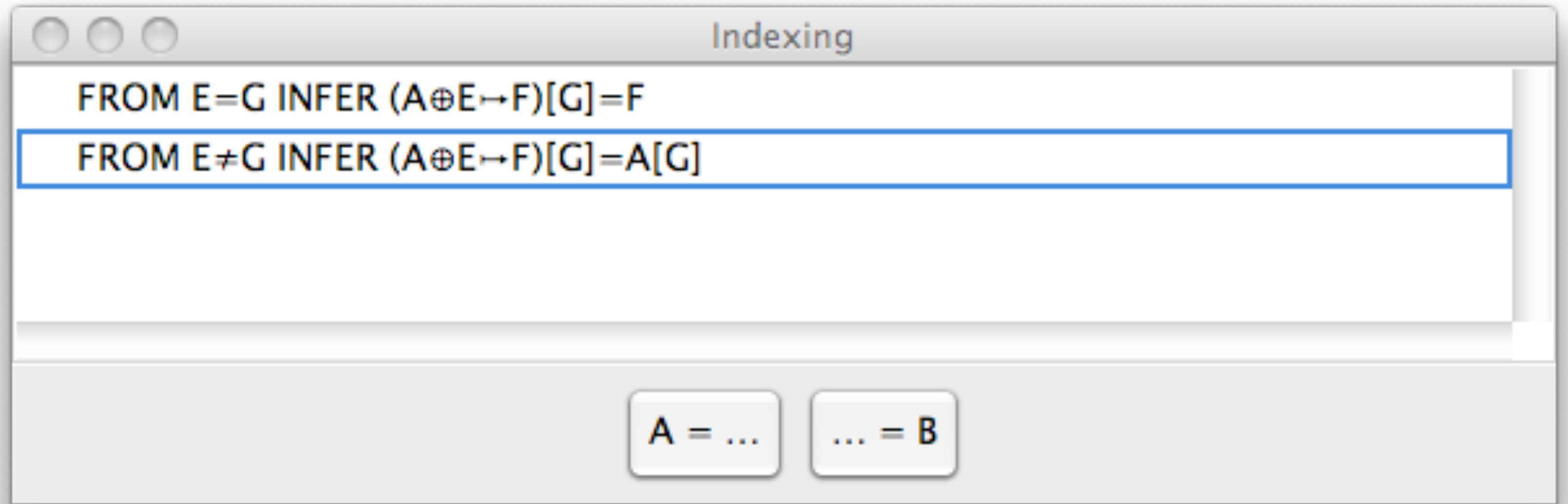
Simplifying Array Indexing Expressions

- JAPE Provides two equality rules for simplifying indexing expressions.

- These are needed for proving just about any array program. They are in the Indexing menu:



Simplifying Array Indexing Expressions



- The first rule deals with the case where the index G on the "new" array is the **same** as the index E that was modified.

- The second deals with the case where they are **not** the same.

- One of these cases needs to be established to perform any simplification.

Simplification Example

- Prove

$$\{a[0] = 98 \land a[1] = 5\}$$
$$a[1] := 99$$
$$\{a[0] = 98 \land a[1] > 50\}$$

...

1: $\{a[0]=98 \land a[1]=5\}(a[1]:=99)$
$\{a[0]=98 \land a[1]>50\}$

Candidates for Simplification

...

1: $a[0]=98 \land a[1]=5$
$\rightarrow \boxed{(a \oplus 1 \mapsto 99)[0]} = 98 \land \boxed{(a \oplus 1 \mapsto 99)[1]} > 50 \land 0 \leq 1 \land 1 < length(a)$

2: $\{(a \oplus 1 \mapsto 99)[0]=98 \land (a \oplus 1 \mapsto 99)[1]>50 \land 0 \leq 1 \land 1 < length(a)\}$ array-element-assignment
$(a[1]:=99)\{a[0]=98 \land a[1]>50\}$

3: $\{a[0]=98 \land a[1]=5\}(a[1]:=99)\{a[0]=98 \land a[1]>50\}$ consequence(L) 1,2

Using Indexing Rule where E≠G

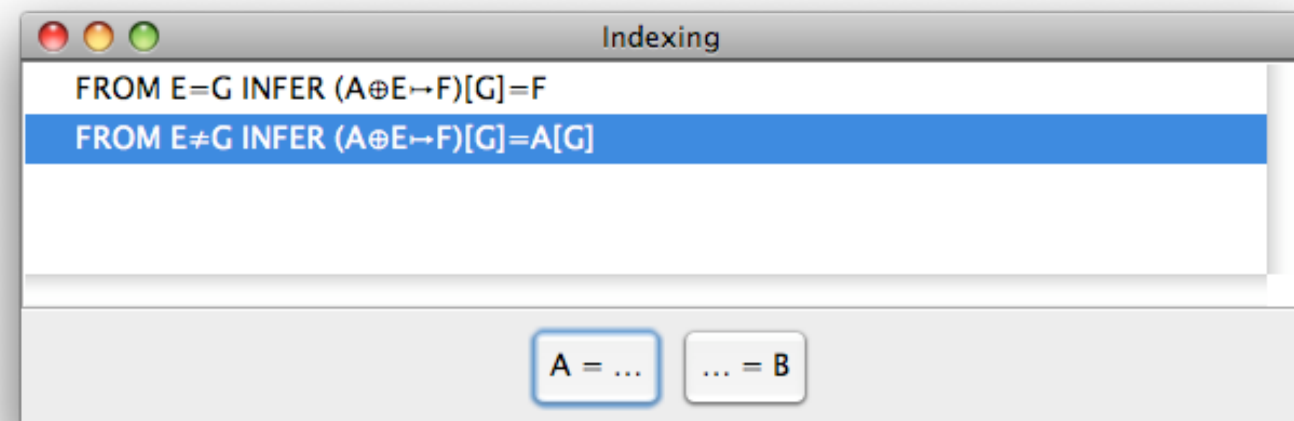
Note that the entire indexed expression must be selected, not just the array portion

...

1: $a[0]=98 \wedge a[1]=5 \rightarrow \boxed{(a \oplus 1 \mapsto 99)[0]}=98 \wedge (a \oplus 1 \mapsto 99)[1]>50 \wedge 0 \leq 1 \wedge 1 < length(a)$

2: $\{(a \oplus 1 \mapsto 99)[0]=98 \wedge (a \oplus 1 \mapsto 99)[1]>50 \wedge 0 \leq 1 \wedge 1 < length(a)\}$ $(a[1]:=99)\{a[0]=98 \wedge a[1]>50\}$ array-element-assignment

3: $\{a[0]=98 \wedge a[1]=5\}(a[1]:=99)\{a[0]=98 \wedge a[1]>50\}$ consequence(L) 1,2



Result

when $0 \neq 1$, $(a \oplus 1 \rightarrow 99)[0]$ is same as $a[0]$ whatever that happens to be.

...

1: $1 \neq 0$

...

2: $a[0]=98 \wedge a[1]=5 \rightarrow \boxed{a[0]}=98 \wedge (a \oplus 1 \mapsto 99)[1]>50 \wedge 0 \leq 1 \wedge 1<\mathrm{length}(a)$

3: $a[0]=98 \wedge a[1]=5$
$\rightarrow (a \oplus 1 \mapsto 99)[0]=98 \wedge (a \oplus 1 \mapsto 99)[1]>50 \wedge 0 \leq 1 \wedge 1<\mathrm{length}(a)$ FROM E≠G INFER... 1,2

4: $\{(a \oplus 1 \mapsto 99)[0]=98 \wedge (a \oplus 1 \mapsto 99)[1]>50 \wedge 0 \leq 1 \wedge 1<\mathrm{length}(a)\}$
$(a[1]:=99)\{a[0]=98 \wedge a[1]>50\}$ array-element-assi...

5: $\{a[0]=98 \wedge a[1]=5\}(a[1]:=99)\{a[0]=98 \wedge a[1]>50\}$ consequence(L) 3,4

Using Indexing Rule where E=G

...

1: $1 \neq 0$

...

2: $a[0]=98 \wedge a[1]=5 \rightarrow a[0]=98 \wedge (a \oplus 1 \mapsto 99)[1] > 50 \wedge 0 \leq 1 \wedge 1 < \mathrm{length}(a)$

3: $a[0]=98 \wedge a[1]=5 \rightarrow (a \oplus 1 \mapsto 99)[0]=98 \wedge (a \oplus 1 \mapsto 99)[1] > 50 \wedge 0 \leq 1 \wedge 1 < \mathrm{length}(a)$ FROM E≠G INFER... 1,2

4: $\{(a \oplus 1 \mapsto 99)[0]=98 \wedge (a \oplus 1 \mapsto 99)[1] > 50 \wedge 0 \leq 1 \wedge 1 < \mathrm{length}(a)\}$ $(a[1]:=99)\{a[0]=98 \wedge a[1] > 50\}$ array-element-assi...

5: $\{a[0]=98 \wedge a[1]=5\}(a[1]:=99)\{a[0]=98 \wedge a[1] > 50\}$ consequence(L) 3,4

Result

where $1=1$, $(a\oplus 1\rightarrow 99)[1]$ is same as RHS of assignment

...

1: $1 \neq 0$

...

2: $a[0]=98 \wedge a[1]=5 \rightarrow a[0]=98 \wedge \boxed{99} > 50 \wedge 0 \leq 1 \wedge 1 < length(a)$

3: $a[0]=98 \wedge a[1]=5 \rightarrow a[0]=98 \wedge (a\oplus 1 \mapsto 99)[1] > 50 \wedge 0 \leq 1 \wedge 1 < length(a)$ FROM E=G INFER (... 2

4: $a[0]=98 \wedge a[1]=5$
$\rightarrow (a\oplus 1 \mapsto 99)[0]=98 \wedge (a\oplus 1 \mapsto 99)[1] > 50 \wedge 0 \leq 1 \wedge 1 < length(a)$ FROM E≠G INFER... 1,3

5: $\{(a\oplus 1 \mapsto 99)[0]=98 \wedge (a\oplus 1 \mapsto 99)[1] > 50 \wedge 0 \leq 1 \wedge 1 < length(a)\}$
$(a[1]:=99)\{a[0]=98 \wedge a[1] > 50\}$ array-element-assi...

6: $\{a[0]=98 \wedge a[1]=5\}(a[1]:=99)\{a[0]=98 \wedge a[1] > 50\}$ consequence(L) 4,5

Using Bounds Inference

Once an index is used, it is assumed to be valid.

Select only indexed expression, not entire statement

1: $a[i]=2$

...

2: $a[i]+1=3 \wedge 0 \leq i \wedge i < length(a)$

Rule

Extras	Window	Help
A=A		
A = ... = B		
obviously		
boundedness from (in)equality		

3: $a[i]=2 \rightarrow a[i]+1=3 \wedge 0 \leq i \wedge i < length(a)$ → intro 1-2

4: $a[i]=2 \rightarrow (a \oplus i \mapsto a[i]+1)[i]=3 \wedge 0 \leq i \wedge i < length(a)$ FROM E=G INFER $(A \oplus E \mapsto F)[G]=F$ 3

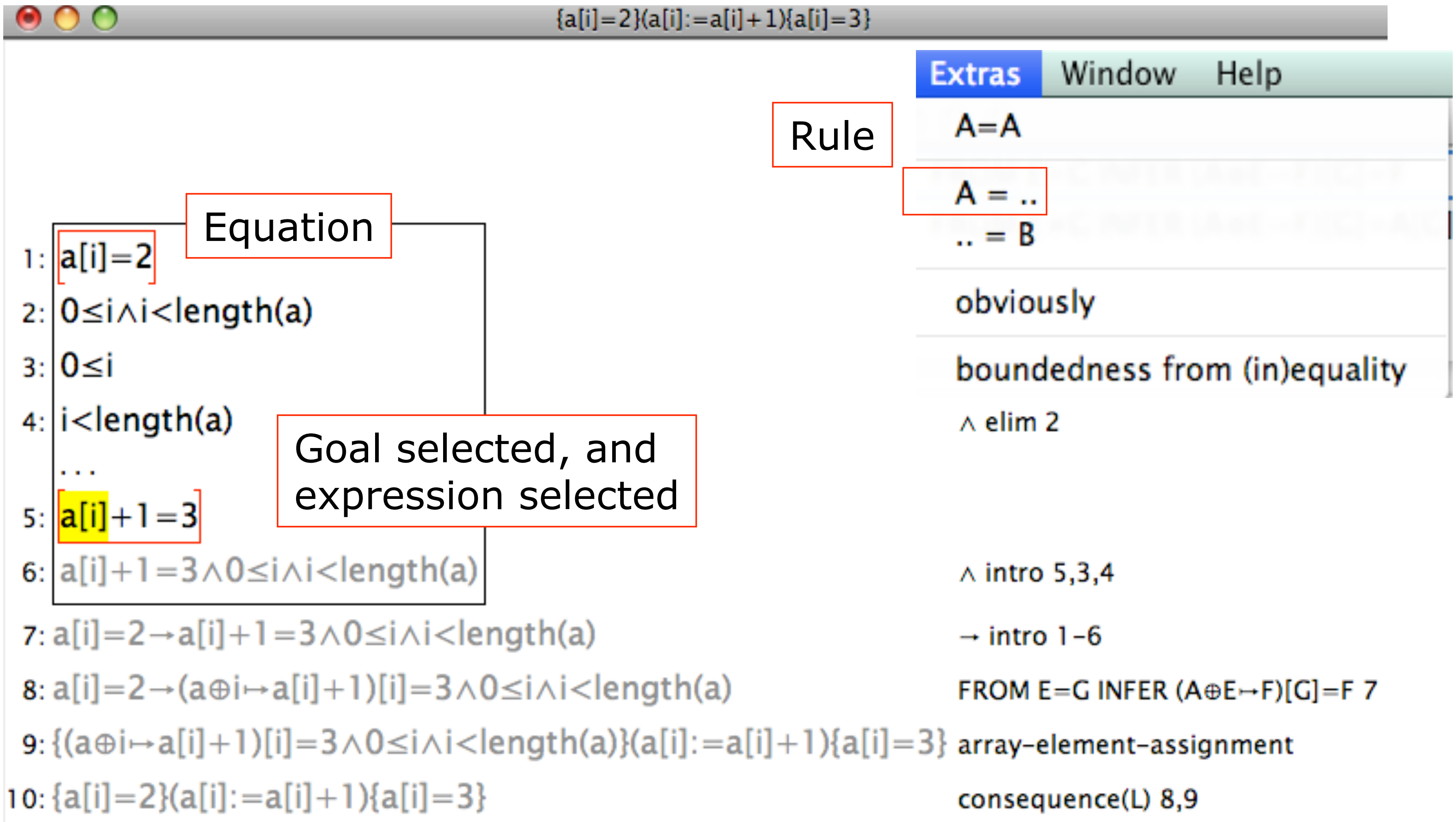
5: $\{(a \oplus i \mapsto a[i]+1)[i]=3 \wedge 0 \leq i \wedge i < length(a)\}(a[i]:=a[i]+1)\{a[i]=3\}$ array-element-assignment

6: $\{a[i]=2\}(a[i]:=a[i]+1)\{a[i]=3\}$ consequence(L) 4,5

Result

1:	$a[i]=2$	assumption
2:	$0\leq i\wedge i<length(a)$	bounded 1
	...	
3:	$a[i]+1=3\wedge 0\leq i\wedge i<length(a)$	
4:	$a[i]=2\rightarrow a[i]+1=3\wedge 0\leq i\wedge i<length(a)$	$\rightarrow$ intro 1–3
5:	$a[i]=2\rightarrow(a\oplus i\mapsto a[i]+1)[i]=3\wedge 0\leq i\wedge i<length(a)$	FROM $E=G$ INFER $(A\oplus E\mapsto F)[G]=F$ 4
6:	$\{(a\oplus i\mapsto a[i]+1)[i]=3\wedge 0\leq i\wedge i<length(a)\}(a[i]:=a[i]+1)\{a[i]=3\}$	array-element-assignment
7:	$\{a[i]=2\}(a[i]:=a[i]+1)\{a[i]=3\}$	consequence(L) 5,6

Completion of Proof, using Equality (complex)



Result

1:	$\boxed{a[i]=2}$	assumption
2:	$0\le i \wedge i<\mathrm{length}(a)$	bounded 1
3:	$0\le i$	$\wedge$ elim 2
4:	$i<\mathrm{length}(a)$	$\wedge$ elim 2
	. . .	
5:	$\boxed{2}+1=3$ was a[i]	
6:	$a[i]+1=3$	equality-substitution 1,5
7:	$a[i]+1=3 \wedge 0\le i \wedge i<\mathrm{length}(a)$	$\wedge$ intro 6,3,4
8:	$a[i]=2 \rightarrow a[i]+1=3 \wedge 0\le i \wedge i<\mathrm{length}(a)$	$\rightarrow$ intro 1-7
9:	$a[i]=2 \rightarrow (a\oplus i\mapsto a[i]+1)[i]=3 \wedge 0\le i \wedge i<\mathrm{length}(a)$	FROM E=G INFER $(A\oplus E\mapsto F)[G]=F$ 8
10:	$\{(a\oplus i\mapsto a[i]+1)[i]=3 \wedge 0\le i \wedge i<\mathrm{length}(a)\}(a[i]:=a[i]+1)\{a[i]=3\}$	array-element-assignment
11:	$\{a[i]=2\}(a[i]:=a[i]+1)\{a[i]=3\}$	consequence(L) 9,10

Completed Proof

1:	$a[i]=2$	assumption
2:	$0\le i \wedge i<length(a)$	bounded 1
3:	$0\le i$	$\wedge$ elim 2
4:	$i<length(a)$	$\wedge$ elim 2
5:	$2+1=3$	obviously
6:	$a[i]+1=3$	equality-substitution 1,5
7:	$a[i]+1=3\wedge 0\le i\wedge i<length(a)$	$\wedge$ intro 6,3,4
8:	$a[i]=2\rightarrow a[i]+1=3\wedge 0\le i\wedge i<length(a)$	$\rightarrow$ intro 1-7
9:	$a[i]=2\rightarrow (a\oplus i\mapsto a[i]+1)[i]=3\wedge 0\le i\wedge i<length(a)$	FROM $E=G$ INFER $(A\oplus E\mapsto F)[G]=F$ 8
10:	$\{(a\oplus i\mapsto a[i]+1)[i]=3\wedge 0\le i\wedge i<length(a)\}(a[i]:=a[i]+1)\{a[i]=3\}$	array-element-assignment
11:	$\{a[i]=2\}(a[i]:=a[i]+1)\{a[i]=3\}$	consequence(L) 9,10

Consecutive Array Modifications

- Could simplify array-modification expressions as soon as they arise, or
- Wait, and deal with nested expressions.
- The first is probably better.

Consecutive Modification Example

$\{a[i] = 0\}$

$a[i] := a[i] + 1;$

$a[i] := a[i] + 1;$

$\{a[2] = 2\}$

...

1: {a[i]=0}(a[i]:=a[i]+1;a[i]:=a[i]+1){a[i]=2}

...

1: {a[i]=0}(a[i]:=a[i]+1){_B2}

...

2: {_B2}(a[i]:=a[i]+1){a[i]=2}

3: {a[i]=0}(a[i]:=a[i]+1;a[i]:=a[i]+1){a[i]=2} sequence 1,2

Early Simplification

...

1: $\{a[i]=0\}\langle a[i]:=a[i]+1\rangle\{(a\oplus i\mapsto a[i]+1)[i]=2\wedge 0\leq i\wedge i<length(a)\}$

2: $\{(a\oplus i\mapsto a[i]+1)[i]=2\wedge 0\leq i\wedge i<length(a)\}\langle a[i]:=a[i]+1\rangle\{a[i]=2\}$ array-element-assignment

3: $\{a[i]=0\}\langle a[i]:=a[i]+1;a[i]:=a[i]+1\rangle\{a[i]=2\}$ sequence 1,2

...

1: $\{a[i]=0\}\langle a[i]:=a[i]+1\rangle\{a[i]+1=2\wedge 0\leq i\wedge i<length(a)\}$

2: $\{a[i]=0\}\langle a[i]:=a[i]+1\rangle\{(a\oplus i\mapsto a[i]+1)[i]=2\wedge 0\leq i\wedge i<length(a)\}$ FROM E=G INFER $(A\oplus E\mapsto F)[G]=F$ 1

3: $\{(a\oplus i\mapsto a[i]+1)[i]=2\wedge 0\leq i\wedge i<length(a)\}\langle a[i]:=a[i]+1\rangle\{a[i]=2\}$ array-element-assignment

4: $\{a[i]=0\}\langle a[i]:=a[i]+1;a[i]:=a[i]+1\rangle\{a[i]=2\}$ sequence 2,3

Early Simplification, step 2

...

1:	$a[i]=0 \rightarrow (a \oplus i \mapsto a[i]+1)[i]+1=2 \wedge 0 \le i \wedge i < length(a)$	
2:	$\{(a \oplus i \mapsto a[i]+1)[i]+1=2 \wedge 0 \le i \wedge i < length(a)\}$ $(a[i]:=a[i]+1)\{a[i]+1=2 \wedge 0 \le i \wedge i < length(a)\}$	array-element-assignment
3:	$\{a[i]=0\}(a[i]:=a[i]+1)\{a[i]+1=2 \wedge 0 \le i \wedge i < length(a)\}$	consequence(L) 1,2
4:	$\{a[i]=0\}(a[i]:=a[i]+1)\{(a \oplus i \mapsto a[i]+1)[i]=2 \wedge 0 \le i \wedge i < length(a)\}$	FROM E=G INFER $(A \oplus E \mapsto F)[G]=F$ 3
5:	$\{(a \oplus i \mapsto a[i]+1)[i]=2 \wedge 0 \le i \wedge i < length(a)\}(a[i]:=a[i]+1)\{a[i]=2\}$	array-element-assignment
6:	$\{a[i]=0\}(a[i]:=a[i]+1;a[i]:=a[i]+1)\{a[i]=2\}$	sequence 4,5

...

1:	$a[i]=0 \rightarrow a[i]+1+1=2 \wedge 0 \le i \wedge i < length(a)$	
2:	$a[i]=0 \rightarrow (a \oplus i \mapsto a[i]+1)[i]+1=2 \wedge 0 \le i \wedge i < length(a)$	FROM E=G INFER $(A \oplus E \mapsto F)[G]=F$ 1
3:	$\{(a \oplus i \mapsto a[i]+1)[i]+1=2 \wedge 0 \le i \wedge i < length(a)\}$ $(a[i]:=a[i]+1)\{a[i]+1=2 \wedge 0 \le i \wedge i < length(a)\}$	array-element-assignment
4:	$\{a[i]=0\}(a[i]:=a[i]+1)\{a[i]+1=2 \wedge 0 \le i \wedge i < length(a)\}$	consequence(L) 2,3
5:	$\{a[i]=0\}(a[i]:=a[i]+1)\{(a \oplus i \mapsto a[i]+1)[i]=2 \wedge 0 \le i \wedge i < length(a)\}$	FROM E=G INFER $(A \oplus E \mapsto F)[G]=F$ 4
6:	$\{(a \oplus i \mapsto a[i]+1)[i]=2 \wedge 0 \le i \wedge i < length(a)\}(a[i]:=a[i]+1)\{a[i]=2\}$	array-element-assignment
7:	$\{a[i]=0\}(a[i]:=a[i]+1;a[i]:=a[i]+1)\{a[i]=2\}$	sequence 5,6

Completed Proof

1:	$a[i]=0$	assumption
2:	$0 \leq i \wedge i < \mathrm{length}(a)$	bounded 1
3:	$0 \leq i$	$\wedge$ elim 2
4:	$i < \mathrm{length}(a)$	$\wedge$ elim 2
5:	$0+1+1=2$	obviously
6:	$a[i]+1+1=2$	equality-substitution 1,5
7:	$a[i]+1+1=2 \wedge 0 \leq i \wedge i < \mathrm{length}(a)$	$\wedge$ intro 6,3,4
8:	$a[i]=0 \rightarrow a[i]+1+1=2 \wedge 0 \leq i \wedge i < \mathrm{length}(a)$	$\rightarrow$ intro 1-7
9:	$a[i]=0 \rightarrow (a \oplus i \mapsto a[i]+1)[i]+1=2 \wedge 0 \leq i \wedge i < \mathrm{length}(a)$	FROM $E=G$ INFER $(A \oplus E \mapsto F)[G]=F$ 8
10:	$\{(a \oplus i \mapsto a[i]+1)[i]+1=2 \wedge 0 \leq i \wedge i < \mathrm{length}(a)\}$ $(a[i]:=a[i]+1)\{a[i]+1=2 \wedge 0 \leq i \wedge i < \mathrm{length}(a)\}$	array-element-assignment
11:	$\{a[i]=0\}(a[i]:=a[i]+1)\{a[i]+1=2 \wedge 0 \leq i \wedge i < \mathrm{length}(a)\}$	consequence(L) 9,10
12:	$\{a[i]=0\}(a[i]:=a[i]+1)\{(a \oplus i \mapsto a[i]+1)[i]=2 \wedge 0 \leq i \wedge i < \mathrm{length}(a)\}$	FROM $E=G$ INFER $(A \oplus E \mapsto F)[G]=F$ 11
13:	$\{(a \oplus i \mapsto a[i]+1)[i]=2 \wedge 0 \leq i \wedge i < \mathrm{length}(a)\}(a[i]:=a[i]+1)\{a[i]=2\}$	array-element-assignment
14:	$\{a[i]=0\}(a[i]:=a[i]+1;a[i]:=a[i]+1)\{a[i]=2\}$	sequence 12,13

Deferred Simplification Alternative

- Use sequence, then array-assignment twice (from the bottom up) to get to this point:

$\{a[i]=0\}(a[i]:=a[i]+1;a[i]:=a[i]+1)\{a[i]=2\}$ [1]

...

messy

1: $a[i]=0 \rightarrow \boxed{(a \oplus i \mapsto a[i]+1 \oplus i \mapsto (a \oplus i \mapsto a[i]+1)[i]+1)[i]}=2 \land 0 \le i \land i < length(a)$

2: $\{(a \oplus i \mapsto a[i]+1 \oplus i \mapsto (a \oplus i \mapsto a[i]+1)[i]+1)[i]=2 \land 0 \le i \land i < length(a)\}$ $(a[i]:=a[i]+1)\{(a \oplus i \mapsto a[i]+1)[i]=2 \land 0 \le i \land i < length(a)\}$ array-element-assignment

3: $\{a[i]=0\}(a[i]:=a[i]+1)\{(a \oplus i \mapsto a[i]+1)[i]=2 \land 0 \le i \land i < length(a)\}$ consequence(L) 1,2

4: $\{(a \oplus i \mapsto a[i]+1)[i]=2 \land 0 \le i \land i < length(a)\}(a[i]:=a[i]+1)\{a[i]=2\}$ array-element-assignment

5: $\{a[i]=0\}(a[i]:=a[i]+1;a[i]:=a[i]+1)\{a[i]=2\}$ sequence 3,4

Provided:

DISTINCT a, i

Completed Proof, using Deferred Simplification

1:	$a[i]=0$	assumption
2:	$0+1+1=2$	obviously
3:	$a[i]+1+1=2$	equality-substitution 1,2
4:	$(a\oplus i\mapsto a[i]+1)[i]+1=2$	FROM E=G INFER (A⊕E↦F)[G]=F 3
5:	$(a\oplus i\mapsto a[i]+1\oplus i\mapsto (a\oplus i\mapsto a[i]+1)[i]+1)[i]=2$	FROM E=G INFER (A⊕E↦F)[G]=F 4
6:	$0\le i\wedge i<length(a)$	bounded 1
7:	$0\le i$	∧ elim 6
8:	$i<length(a)$	∧ elim 6
9:	$\boxed{(a\oplus i\mapsto a[i]+1\oplus i\mapsto (a\oplus i\mapsto a[i]+1)[i]+1)[i]}=2\wedge 0\le i\wedge i<length(a)$	∧ intro 5,7,8
10:	$a[i]=0\rightarrow (a\oplus i\mapsto a[i]+1\oplus i\mapsto (a\oplus i\mapsto a[i]+1)[i]+1)[i]=2\wedge 0\le i\wedge i<length(a)$	→ intro 1-9
11:	$\{(a\oplus i\mapsto a[i]+1\oplus i\mapsto (a\oplus i\mapsto a[i]+1)[i]+1)[i]=2\wedge 0\le i\wedge i<length(a)\}$ $(a[i]:=a[i]+1)\{(a\oplus i\mapsto a[i]+1)[i]=2\wedge 0\le i\wedge i<length(a)\}$	array-element-assignment
12:	$\{a[i]=0\}(a[i]:=a[i]+1)\{(a\oplus i\mapsto a[i]+1)[i]=2\wedge 0\le i\wedge i<length(a)\}$	consequence(L) 10,11
13:	$\{(a\oplus i\mapsto a[i]+1)[i]=2\wedge 0\le i\wedge i<length(a)\}(a[i]:=a[i]+1)\{a[i]=2\}$	array-element-assignment
14:	$\{a[i]=0\}(\boxed{a[i]:=a[i]+1;a[i]:=a[i]+1})\{a[i]=2\}$	sequence 12,13

original program

Provided:

DISTINCT a, i

Read-Write Example

- Suppose we wish to copy one array **a** into another **b**.

- We'll assume both arrays are the same length.

- We need to assert that they have the same elements in the same positions.

Assumption and Expectation

$\{n=\text{length}(a) \land n=\text{length}(b) \land 0 \leq n\}$

… array-copy program …

$\{\forall j.((0 \leq j \land j<n) \rightarrow (b[j]=a[j]))\}$

assuming indices run 0, … n-1

Proposed Program

$\{n=length(a) \land n=length(b) \land 0 \le n\}$

```
(i := 0)

while i < n
    do
    b[i] := a[i];
    i := i+1
    od
```

$\{\forall j.((0 \le j \land j<n) \rightarrow (b[j]=a[j]))\}$

One would hope for a short proof, but …

Program with Invariant (for JAPE)

WHERE DISTINCT n,a,b,i IS

$\{n=length(a) \land n=length(b) \land 0 \le n\}$

(i := 0)

$\{n=length(a) \land n=length(b) \land 0 \le n \land 0 \le i \land i \le n$
$\land \forall j.((0 \le j \land j<i) \rightarrow (b[j]=a[j]))\}$

while i < n do b[i] := a[i]; i := i+1 od

$\{\forall j.((0 \le j \land j<n) \rightarrow (b[j]=a[j]))\}$

Invariant Part of Loop Body

18:	$n=\mathrm{length}(a)\wedge n=\mathrm{length}(b)\wedge 0\le n\wedge 0\le i\wedge i\le n\wedge \forall j.((0\le j\wedge j<i)\to(b[j]=a[j]))\wedge i<n$	assumption
19:	$n=\mathrm{length}(a)\wedge n=\mathrm{length}(b)\wedge 0\le n\wedge 0\le i+1\wedge i+1\le n\wedge \forall j.((0\le j\wedge j<i+1)\to((b\oplus i\mapsto a[i])[j]=a[j]))\wedge 0\le i\wedge i<\mathrm{length}(b)\wedge i<\mathrm{length}(a)$	{cut}
20:	$n=\mathrm{length}(a)\wedge n=\mathrm{length}(b)\wedge 0\le n\wedge 0\le i\wedge i\le n\wedge \forall j.((0\le j\wedge j<i)\to(b[j]=a[j]))\wedge i<n$ $\to n=\mathrm{length}(a)\wedge n=\mathrm{length}(b)\wedge 0\le n\wedge 0\le i+1\wedge i+1\le n\wedge \forall j.((0\le j\wedge j<i+1)\to((b\oplus i\mapsto a[i])[j]=a[j]))\wedge 0\le i\wedge i<\mathrm{length}(b)\wedge i<\mathrm{length}(a)$	$\to$ intro 18-19
21:	$\{n=\mathrm{length}(a)\wedge n=\mathrm{length}(b)\wedge 0\le n\wedge 0\le i+1\wedge i+1\le n\wedge \forall j.((0\le j\wedge j<i+1)\to((b\oplus i\mapsto a[i])[j]=a[j]))\wedge 0\le i\wedge i<\mathrm{length}(b)\wedge i<\mathrm{length}(a)\}$ $(b[i]:=a[i])\{n=\mathrm{length}(a)\wedge n=\mathrm{length}(b)\wedge 0\le n\wedge 0\le i+1\wedge i+1\le n\wedge \forall j.((0\le j\wedge j<i+1)\to(b[j]=a[j]))\}$	array-element-assignment
22:	$\{n=\mathrm{length}(a)\wedge n=\mathrm{length}(b)\wedge 0\le n\wedge 0\le i\wedge i\le n\wedge \forall j.((0\le j\wedge j<i)\to(b[j]=a[j]))\wedge i<n\}(b[i]:=a[i])$ $\{n=\mathrm{length}(a)\wedge n=\mathrm{length}(b)\wedge 0\le n\wedge 0\le i+1\wedge i+1\le n\wedge \forall j.((0\le j\wedge j<i+1)\to(b[j]=a[j]))\}$	consequence(L) 20,21
23:	$\{n=\mathrm{length}(a)\wedge n=\mathrm{length}(b)\wedge 0\le n\wedge 0\le i+1\wedge i+1\le n\wedge \forall j.((0\le j\wedge j<i+1)\to(b[j]=a[j]))\}$ $(i:=i+1)\{n=\mathrm{length}(a)\wedge n=\mathrm{length}(b)\wedge 0\le n\wedge 0\le i\wedge i\le n\wedge \forall j.((0\le j\wedge j<i)\to(b[j]=a[j]))\}$	variable-assignment
24:	$\{n=\mathrm{length}(a)\wedge n=\mathrm{length}(b)\wedge 0\le n\wedge 0\le i\wedge i\le n\wedge \forall j.((0\le j\wedge j<i)\to(b[j]=a[j]))\wedge i<n\}$ $(b[i]:=a[i];i:=i+1)\{n=\mathrm{length}(a)\wedge n=\mathrm{length}(b)\wedge 0\le n\wedge 0\le i\wedge i\le n\wedge \forall j.((0\le j\wedge j<i)\to(b[j]=a[j]))\}$	sequence 22,23

Details Collapsed on Previous Slide

18:	$n=\mathrm{length}(a) \wedge n=\mathrm{length}(b) \wedge 0 \le n \wedge 0 \le i \wedge i \le n \wedge \forall j.((0 \le j \wedge j<i) \rightarrow (b[j]=a[j])) \wedge i<n$	assumption
19:	$n=\mathrm{length}(a)$	$\wedge$ elim 18
20:	$n=\mathrm{length}(b)$	$\wedge$ elim 18
21:	$0 \le n$	$\wedge$ elim 18
22:	$0 \le i$	$\wedge$ elim 18
23:	$\forall j.((0 \le j \wedge j<i) \rightarrow (b[j]=a[j]))$	$\wedge$ elim 18
24:	$i<n$	$\wedge$ elim 18
25:	$0 \le i+1$	$A \le B \vdash A \le B+1$ 22
26:	$i+1 \le n$	$A+1 \le B \triangleq A<B$ 24
27:	integer i3	assumption
28:	$0 \le i3 \wedge i3<i+1$	assumption
29:	$0 \le i3$	$\wedge$ elim 28
30:	$i3<i+1$	$\wedge$ elim 28
31:	$i3<i \vee i3=i$	$A<B+1 \vdash A<B \vee A=B$ 30
32:	$i3<i$	assumption
33:	$0 \le i3 \wedge i3<i$	$\wedge$ intro 29,32
34:	$(0 \le i3 \wedge i3<i) \rightarrow (b[i3]=a[i3])$	$\forall$ elim 23
35:	$b[i3]=a[i3]$	$\rightarrow$ elim 34,33
36:	$i3 \ne i$	$A<B \vdash A \ne B$ 32
37:	$i \ne i3$	$A \ne B \triangleq B \ne A$ 36
38:	$(b \oplus i \mapsto a[i])[i3]=a[i3]$	FROM $E \ne G$ INFER $(A \oplus E \mapsto F)[G]=\ldots$ 37,35
39:	$i3=i$	assumption
40:	$i=i3$	$A=B \triangleq B=A$ 39
41:	$a[i]=a[i3]$	$i=j \vdash a[i]=a[j]$ 40
42:	$(b \oplus i \mapsto a[i])[i3]=a[i3]$	FROM $E=G$ INFER $(A \oplus E \mapsto F)[G]=F$ 40,41
43:	$(b \oplus i \mapsto a[i])[i3]=a[i3]$	$\vee$ elim 31,32-38,39-42
44:	$(0 \le i3 \wedge i3<i+1) \rightarrow ((b \oplus i \mapsto a[i])[i3]=a[i3])$	$\rightarrow$ intro 28-43
45:	$\forall j.((0 \le j \wedge j<i+1) \rightarrow ((b \oplus i \mapsto a[i])[j]=a[j]))$	$\forall$ intro 27-44
46:	$i<\mathrm{length}(b)$	$i<n,\ n=\mathrm{length}(a) \vdash i<\mathrm{length}(a)$ 24,20
47:	$i<\mathrm{length}(a)$	$i<n,\ n=\mathrm{length}(a) \vdash i<\mathrm{length}(a)$ 24,19
48:	$n=\mathrm{length}(a) \wedge n=\mathrm{length}(b) \wedge 0 \le n \wedge 0 \le i+1 \wedge i+1 \le n \wedge \forall j.((0 \le j \wedge j<i+1) \rightarrow ((b \oplus i \mapsto a[i])[j]=a[j])) \wedge 0 \le i \wedge i<\mathrm{length}(b) \wedge i<\mathrm{length}(a)$	$\wedge$ intro 19,20,21,25,26,45,22,46,47

Exercises

- Provide assumption and expectation specifications for a sorting program, then prove its total correctness.

- Work through the minimal sum-section (Huth&Ryan, Example 4.19). Note the implications for algorithmic efficiency (linear for the given method vs. $O(n^3)$ for the naïve method).