Sequent Calculus

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Sequent Calculus (SC) (Gentzen System S)

- The **sequent calculus** was used by Gentzen to derive results about Natural Deduction. It allows **sets** of formulas on **both** sides of |–.
- In the SC, |– becomes an **object-language** symbol, rather than a meta-language symbol as we have been using it. This is similar to the **contextual representation** used in the soundness proof. However, some authors use ⇒ or → instead of |–.
- The *intuitive* meaning of |– is:

$A_1, \ldots, A_m \vdash B_1, \ldots, B_n$ is like

$(A_1 \wedge \ldots \wedge A_m) \rightarrow (B_1 \vee \ldots \vee B_n)$ [yes, ∨ **on the right!**] which is analogous to

$\neg A_1 \vee \ldots \vee \neg A_m \vee B_1 \vee \ldots \vee B_n$

Thus one needs to pay careful attention to left vs. right.

The case of n = 1 is typically of most interest for the last step.

One way to think about n > 1 is that there a proof of one of n distinct formulas.

Other Presentations of SC

- Smullyan and Prawitz use $\rightarrow$ instead of |–, but use $\supset$ for implication.
- Ben-Ari uses $\Rightarrow$ instead of |–
  to emphasize that, in SC, $\Rightarrow$ is part of the object language.
- Some other authors and JAPE use |–.

Relation to Tableaux

- This presentation is based on classical logic.
- Intuitionistic proofs are a special case with only one formula on the right-hand sides.
- There is a strong connection between SC and block tableaux proofs, as we later discuss.
- Tableaux are based on refutation. But SC provides a corresponding proof without using refutation.

Sequent Calculus Rules

- Instead of introduction and elimination, as in natural deduction, SC uses introduction on ***both*** sides:

  introduction on the right of |–

  introduction on the left of |–

- Proofs are most easily constructed as a tree, ***upward*** from the goal toward "axioms". However, the word "introduction" is used as if going downward.

- Take note of which rules "branch" the tree going upward.

- In what follows, ____ and … stand for ***sets*** of formulas that are **identified** above and below the line.

SC Axiom

- In SC, there is one axiom (rule with no antecedent):

$$\frac{}{\_\_\_P\_\_ \mid\!- \ldots P \ldots}\ \text{Axiom}$$

- Here ___ and … represent sets of other formulas.

- The meaning in this case is that any formula can be derived from itself.

∧ Rules

- |–∧ means introducing ∧ on the right (going downward)

$$\frac{\_\_\_ \vdash \ldots A \ldots \qquad \_\_\_ \vdash \ldots B \ldots}{\_\_\_ \vdash \ldots A \wedge B \ldots} \vdash\wedge$$

branching

meaning: to **derive** A ∧ B, it suffices to derive A and B separately (similar to ∧ Introduction).

- ∧|– means introducing ∧ on the left

$$\frac{\_\_\_ \ A, B \ \_\_\_ \vdash \ldots}{\_\_\_ \ A \wedge B \ \_\_\_ \vdash \ldots} \wedge\vdash$$

meaning: to **use** A and B individually can use A ∧ B (similar to ∧ Elimination).

∨ Rules

___ |– … A, B …

____ |–∨

___ |– … A ∨ B …

meaning: to **derive** A ∨ B, it suffices to derive one or the other
(similar to ∨ Introduction).

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branching

___ A ___ |– … ___ B ___ |– …

____ ∨|–

___ A ∨ B ___ |– …

meaning: to **use** A ∨ B, we need separate derivations from A vs. B
(similar to ∨ Elmination).

→ Rules

$$\frac{\_\_\_\ A\ \_\_\_\ \vdash \ldots B \ldots}{\_\_\_\ \vdash \ldots A \rightarrow B \ldots}\ \vdash\rightarrow$$

meaning: to **derive** A → B, it suffices to derive B with A as an added assumption (similar to → Introduction).

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branching

$$\frac{\_\_\_\ \vdash \ldots A \ldots \qquad \_\_\_\ B\ \_\_\_\ \vdash \ldots}{\_\_\_\ A \rightarrow B\ \_\_\_\ \vdash \ldots}\ \rightarrow\vdash$$

meaning: If A can be proved, and we can achieve the proof an ultimate result (…) based on B as an added assumption,
then the ultimate result can be proved by assuming A → B.
(similar to → Elimination).

¬ Rules

___ A ___ |– …

____ |–¬

___ |– …¬A …

meaning: to derive ¬A … from ___, it suffices to derive … from A ___ (similar to ¬ Introduction).

___ |– … A …

____ ¬|–

___ ¬A ___ |– …

meaning: to derive … from ¬A ___, it suffices to derive A … from ___ (similar to RAA).

SC Rule Summary

	Introduce on Left	Introduce on Right
∧	___ A, B ___ |– … ―――――――――――― ∧|– ___ A ∧ B ___ |– …	___ |– … A … ___ |– … B … ―――――――――――― |–∧ ___ |– … A ∧ B …
∨	___ A ___ |– … ___ B ___ |– … ―――――――――――― ∨|– ___ A ∨ B ___ |– …	___ |– … A, B … ―――――――――――― |–∨ ___ |– … A ∨ B …
→	___ |– … A … ___ B ___ |– … ―――――――――――― →|– ___ A → B ___ |– …	___ A ___ |– … B … ―――――――――――― |–→ ___ |– … A → B …
¬	___ |– … A … ―――――――――――― ¬|– ___ ¬A ___ |– …	___ A ___ |– … ―――――――――――― |–¬ ___ |– … ¬A …

Sequent Calculus Rules

(Ben-Ari version for comparison)

Definition 3.46 Axioms in the Gentzen sequent system $\mathcal{S}$ are sequents of the form: $U \cup \{A\} \Rightarrow V \cup \{A\}$. The rules of inference are:

op	Introduction into consequent	Introduction into antecedent
$\wedge$	$\dfrac{U \Rightarrow V \cup \{A\} \qquad U \Rightarrow V \cup \{B\}}{U \Rightarrow V \cup \{A \wedge B\}}$	$\dfrac{U \cup \{A, B\} \Rightarrow V}{U \cup \{A \wedge B\} \Rightarrow V}$
$\vee$	$\dfrac{U \Rightarrow V \cup \{A, B\}}{U \Rightarrow V \cup \{A \vee B\}}$	$\dfrac{U \cup \{A\} \Rightarrow V \qquad U \cup \{B\} \Rightarrow V}{U \cup \{A \vee B\} \Rightarrow V}$
$\rightarrow$	$\dfrac{U \cup \{A\} \Rightarrow V \cup \{B\}}{U \Rightarrow V \cup \{A \rightarrow B\}}$	$\dfrac{U \Rightarrow V \cup \{A\} \qquad U \cup \{B\} \Rightarrow V}{U \cup \{A \rightarrow B\} \Rightarrow V}$
$\neg$	$\dfrac{U \cup \{A\} \Rightarrow V}{U \Rightarrow V \cup \{\neg A\}}$	$\dfrac{U \Rightarrow V \cup \{A\}}{U \cup \{\neg A\} \Rightarrow V}$

Sequent Calculus Strategy Summary

Working **backward/upward** from the desired sequent:

Situation	Action
A on left and right	Axiom.
$\neg A$ in a right formula	Replace with A on the left.
$\neg A$ in a left formula	Replace with A on the right.
$A \wedge B$ in a right formula	**Split** right set into A and B versions.
$A \wedge B$ in a left formula	Replace the formula with A, B.
$A \vee B$ in a right formula	Replace the formula with A, B.
$A \vee B$ in a left formula	**Split** left set into A and B versions.
$A \rightarrow B$ in a right formula	Replace with A on the left, B on the right.
$A \rightarrow B$ in a left formula	**Split** into two versions with A on the right in one, B on the left in the other.

Example Sequent Calculus Proof

Constructed from bottom to top:

$P \vee Q, \neg P \vdash Q$

Example Sequent Calculus Proof

Constructed from bottom to top:

$$\frac{P \vee Q \ \vdash P, Q}{P \vee Q, \neg P \vdash Q} \qquad \neg\vdash \text{ rule}$$

Example Sequent Calculus Proof

Constructed from bottom to top:

P |– P, Q　　　　　　Q, |– P, Q　　　　v|– rule

P v Q |– P, Q　　　　¬|– rule

P v Q, ¬P |–Q

Example Sequent Calculus Proof

Constructed from bottom to top:

$$\dfrac{\dfrac{\dfrac{}{P \vdash P, Q}\ \text{Axiom} \qquad \dfrac{}{Q, \vdash P, Q}\ \text{Axiom}}{P \lor Q \vdash P, Q}\ \lor\vdash \text{ rule}}{P \lor Q, \neg P \vdash Q}\ \neg\vdash \text{ rule}$$

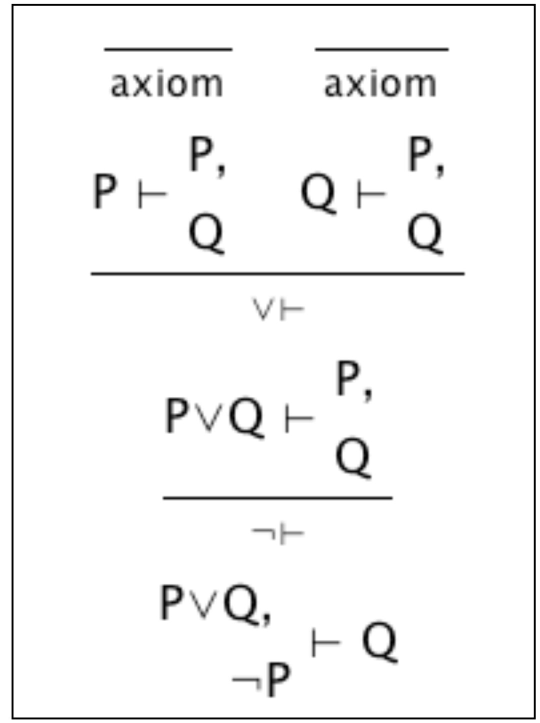
Compare JAPE Version (MCS)

_________ Axiom   _________ Axiom

P |–P, Q   Q, |– P, Q   v |– rule

P v Q |– P, Q   ¬ |– rule

P v Q, ¬P |–Q



MCS = Multi-Conclusion (classical)
SCS = Single Conclusion (intuitionistic)

More Sequent Calculus Examples
(constructed working backward/upward)

$\vdash ((P \rightarrow Q) \wedge (Q \rightarrow R)) \rightarrow (P \rightarrow R)$

More Sequent Calculus Examples

$$\frac{(P \rightarrow Q) \wedge (Q \rightarrow R) \;\; |\!- \;\; (P \rightarrow R)}{|\!- \; ((P \rightarrow Q) \wedge (Q \rightarrow R)) \rightarrow (P \rightarrow R)} \quad |\!- \rightarrow$$

More Sequent Calculus Examples

$$\frac{\dfrac{(P \to Q), (Q \to R) \;\vdash\; (P \to R)}{(P \to Q) \wedge (Q \to R) \;\vdash\; (P \to R)}\;\wedge\vdash}{\vdash ((P \to Q) \wedge (Q \to R)) \to (P \to R)}\;\vdash\to$$

More Sequent Calculus Examples

$\underline{(P \to Q), (Q \to R), P \;\; \vdash \;\; R}$	$\vdash \to$
$\underline{(P \to Q), (Q \to R) \;\; \vdash \;\; (P \to R)}$	$\wedge \vdash$
$\underline{(P \to Q) \wedge (Q \to R) \;\; \vdash \;\; (P \to R)}$	$\vdash \to$
$\vdash ((P \to Q) \wedge (Q \to R)) \to (P \to R)$	

More Sequent Calculus Examples

$$\frac{\dfrac{\dfrac{\dfrac{(P \to Q), R, P \vdash R \qquad (P \to Q), P \vdash Q, R}{(P \to Q), (Q \to R), P \vdash R}\ \to\vdash}{(P \to Q), (Q \to R) \vdash (P \to R)}\ \vdash\to}{(P \to Q) \wedge (Q \to R) \vdash (P \to R)}\ \wedge\vdash}{\vdash ((P \to Q) \wedge (Q \to R)) \to (P \to R)}\ \vdash\to$$

More Sequent Calculus Examples

$\overline{\qquad\qquad\qquad}$ Ax $\quad$ $P \vdash Q, R, P \qquad Q, P \vdash Q, R \quad \rightarrow\vdash$

$(P \rightarrow Q), R, P \vdash R \qquad (P \rightarrow Q), P \vdash Q, R \quad \rightarrow\vdash$

$(P \rightarrow Q), (Q \rightarrow R), P \vdash R \quad \vdash\rightarrow$

$(P \rightarrow Q), (Q \rightarrow R) \vdash (P \rightarrow R) \quad \wedge\vdash$

$(P \rightarrow Q) \wedge (Q \rightarrow R) \vdash (P \rightarrow R) \quad \vdash\rightarrow$

$\vdash ((P \rightarrow Q) \wedge (Q \rightarrow R)) \rightarrow (P \rightarrow R)$

More Sequent Calculus Examples

$$\dfrac{\dfrac{}{(P \to Q), R, P \vdash R}\text{Ax} \qquad \dfrac{\dfrac{}{P \vdash Q, R, P}\text{Ax} \qquad \dfrac{}{Q, P \vdash Q, R}\text{Ax}}{(P \to Q), P \vdash Q, R}\to\vdash}{\dfrac{(P \to Q), (Q \to R), P \vdash R}{\dfrac{(P \to Q), (Q \to R) \vdash (P \to R)}{\dfrac{(P \to Q) \wedge (Q \to R) \vdash (P \to R)}{\vdash ((P \to Q) \wedge (Q \to R)) \to (P \to R)}\vdash\to}\wedge\vdash}\vdash\to}\to\vdash$$

JAPE SCS vs. MCS

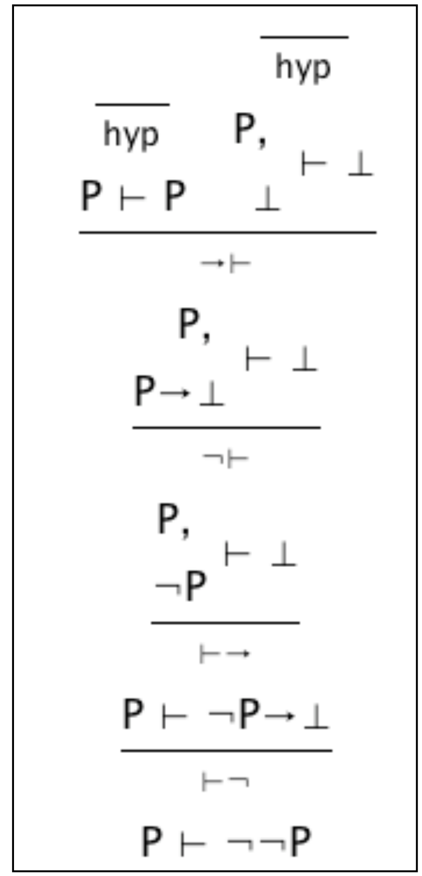
- SCS: Single-conclusion:

  Will only prove intuitionistic sequents.

  The RHS is always a single formula.

  $\neg\varphi$ converts to $\varphi\rightarrow\bot$ first.

  (I'm not sure why.)



This is ¬¬ Introduction.

¬¬ Elimination cannot be proved intuitionistically.

SC vs. Tableaux

- A correspondence can be established between an SC proof and a **block tableau** proof T.
- Tableau is constructed top-down; SC is bottom-up.
- The conclusion of SC is negated in T.
- The premises of SC, if any, are unnegated in T.
- There is a correspondence between rules of the two systems.
- Splitting in SC is like splitting in the block tableau.
- Axioms of SC correspond to path closure.

What are Blocks?

- A block is simply a set of formulas corresponding to the ***un-checked*** formulas on a **single path** in the tree tableau.

- We start with one block.
- **Stacking** replaces formulas **within** that block.
- **Splitting** splits the block into two.
- **Closure** is when a block contains a formula and its negation.

Block Tableau from Tableau Example Negated Formula at Root

1. $\neg((p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p))$

Corresponding Block Tableau

$\{ \neg((p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)) \}$

1. $\neg((p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p))$ ✓

2. $(p \rightarrow q)$

3. $\neg(\neg q \rightarrow \neg p)$

Corresponding Block Tableau

$\{ (p \rightarrow q), \neg(\neg q \rightarrow \neg p) \}$

1. $\neg((p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p))$ ✓

2. $(p \rightarrow q)$

3. $\neg(\neg q \rightarrow \neg p)$ ✓
4. $\neg q$
5. $\neg\neg p$

Corresponding Block Tableau

$\{ (p \rightarrow q), \neg q, \neg\neg p \}$

1. $\neg((p \to q) \to (\neg q \to \neg p))$ ✓

2. $(p \to q)$

3. $\neg(\neg q \to \neg p)$ ✓

4. $\neg q$

5. $\neg\neg p$ ✓

6. p

Corresponding Block Tableau

$\{(p \to q), \neg q, p\}$

1. ¬((p → q) → (¬q → ¬p)) ✓

2. (p → q) ✓

3. ¬(¬q → ¬p) ✓

4. ¬q

5. ¬¬p ✓

6. p

Split from line 2.

7. ¬p — X(6, 7)

8. q — X(4, 8)

Corresponding Block Tableau Splits

{¬p ¬q, p }, {q, ¬q, p } both blocks close

Sequent Calculus vs. Tableaux

Sequent Calculus	**Tableau**
Constructed bottom-up.	Constructed top-down.
Proves $A_1, \ldots, A_m \vdash B_1, \ldots, B_n$ Equivalent to proving $(\neg A_1 \lor \ldots \lor \neg A_m \lor B_1 \lor \ldots \lor B_n)$	Negated conclusion (typically m = 0, n = 1) $\neg(\neg A_1 \lor \ldots \lor \neg A_m \lor B_1 \lor \ldots \lor B_n)$ $\equiv A_1 \land \ldots \land A_m \land \neg B_1 \land \ldots \land \neg B_n$
Formulas on the right	Originally negated formulas
Formulas on the left	Originally un-negated formulas
Axiom …, p, … |- __, p, __	Closure p, ¬p

Tableau vs. SC Example

negated formulas in tableau go on the right of $\vdash$

unnegated formulas in tableau go on the right of $\vdash$

Block Tableau
$\{ \neg((p \to q) \to (\neg q \to \neg p)) \}$

SC "upside-down"
$\vdash (p \to q) \to (\neg q \to \neg p)$

Tableau vs. SC Example

Block Tableau
$\{ \neg((p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)) \}$
$\{ (p \rightarrow q), \neg(\neg q \rightarrow \neg p) \}$

SC "upside-down"
$\vdash (p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$
$(p \rightarrow q) \vdash (\neg q \rightarrow \neg p)$

Tableau vs. SC Example

Block Tableau
$\{ \neg((p \to q) \to (\neg q \to \neg p)) \}$
$\{ (p \to q), \neg(\neg q \to \neg p) \}$
$\{ (p \to q), \neg q, \neg\neg p \}$

SC "upside-down"
$\vdash (p \to q) \to (\neg q \to \neg p)$
$(p \to q) \vdash (\neg q \to \neg p)$
$\neg q, (p \to q) \vdash \neg p$

Tableau vs. SC Example

Block Tableau	SC "upside-down"
{ ¬((p → q) → (¬q → ¬p)) }	|– (p → q) → (¬q → ¬p)
{ (p → q), ¬(¬q → ¬p) }	(p → q) |– (¬q → ¬p)
{ (p → q), ¬q, ¬¬p }	¬q, (p → q) |– ¬p
{ (p → q), ¬q, p }	¬q, (p → q), p |–

Tableau vs. SC Example

Block Tableau	**SC "upside-down"**
{ ¬((p → q) → (¬q → ¬p)) }	|– (p → q) → (¬q → ¬p)
{ (p → q), ¬(¬q → ¬p) }	(p → q) |– (¬q → ¬p)
{ (p → q), ¬q, ¬¬p }	¬q, (p → q) |– ¬p
{ (p → q), ¬q, p }	¬q, (p → q), p |–
{¬p, ¬q, p }X { q, ¬q, p }X	¬q, q, p |– q, p |– p(ax)
	q, p |– q (ax)

Rough Correspondence: SC vs. ND

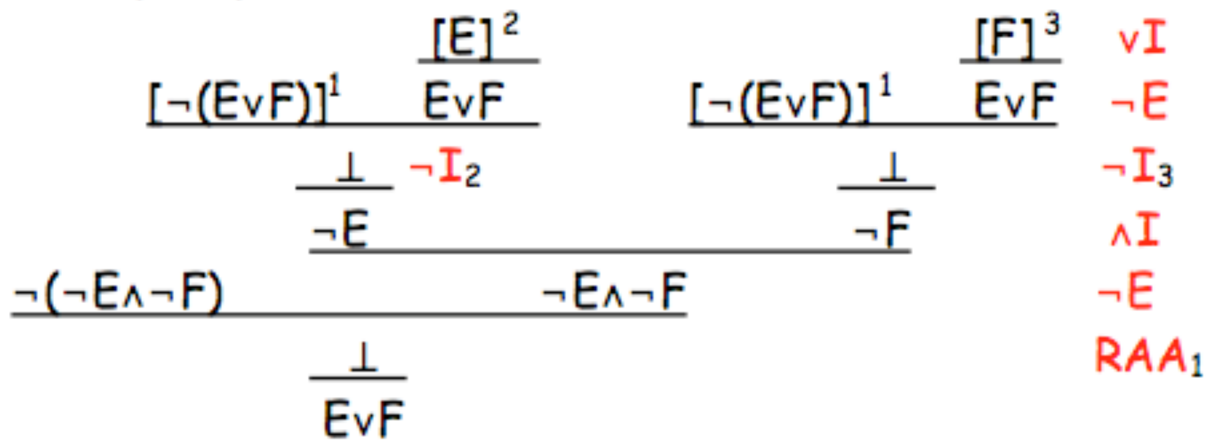
- The last sequent of an SC derivation will always correspond to the overall sequent derived in ND.
- Other sequents may correspond to subproofs.
- Moving from **bottom to top** in an SC proof is like working **outside-in** in a ND proof.
- Consider LHS of a sequent to be *all* operative hypotheses, including assumptions in sub-proofs.
- Consider RHS to be goal (as a disjunction).

Conversion Between SC and ND

- http://twelf.plparty.org/wiki/POPL_Tutorial/Sequent_vs_Natural_Deduction
- http://www.ags.uni-sb.de/~chris/papers/2002-pisa.pdf

One Sequent ¬(¬E∧¬F) |- E∨F
Proved Five Ways

Natural Deduction (tree)



One Sequent ¬(¬E∧¬F) |- E∨F
Proved Five Ways

Natural Deduction (Fitch diagram)

1	¬(¬E∧¬F)	premise
2	¬(E∨F)	assumption
3	E	assumption
4	E∨F	3, ∨I
5	⊥	2, 4, ¬E
6	¬E	3-5, ¬I
7	F	assumption
8	E∨F	7, ∨I
9	⊥	2, 8, ¬E
10	¬F	7-9, ¬I
11	¬E∧¬F	6, 10, ∧I
12	⊥	1, 11, ¬E
13	E∨F	2-12, RAA

One Sequent ¬(¬E∧¬F) |- E∨F Proved Five Ways

Sequent Calculus (read bottom-up)

E |- E, F	F |- E, F	|-¬
|- E, F, ¬E	|- E, F, ¬F	|-∧
|- E, F, (¬E∧¬F)		¬|-
¬(¬E∧¬F) |- E, F		|-∨
¬(¬E∧¬F) |- E∨F		

Here the correspondence with the Natural Deduction proof is not 1-1.

One Sequent $\neg(\neg E \wedge \neg F) \mid\!- E \vee F$ Proved Five Ways

Block Tableau

$\{ \neg(\neg E \wedge \neg F),\ \neg(E \vee F) \}$
$\{ \neg(\neg E \wedge \neg F),\ \neg E, \neg F \}$
$\{ \neg\neg E \vee \neg\neg F,\ \neg E, \neg F \}$

$\{ \neg\neg E, \neg E, \neg F \}$	$\{ \neg\neg F, \neg E, \neg F \}$
$\{ E, \neg E, \neg F \}$	$\{ F, \neg E, \neg F \}$
X	X

The blocks could close one step earlier, due to $\neg\neg E$ and $\neg E$, etc.

One Sequent ¬(¬E∧¬F) |- E∨F
Proved Five Ways

Tableau (tree form)

1. ¬(¬E∧¬F)	premise √		
2. ¬(E∨F)	negated conclusion √		
3. ¬E	(2 stack)		
4. ¬F	(2 stack)		
5. ¬¬E (1 split) √		6. ¬¬F (1 split) √	
7. E (5 ¬¬)		8. F (5¬¬)	
X (3, 7)		X (4, 8)	

The paths could close one step earlier, due to ¬¬E and ¬E, etc.

An Automated Sequent Calculus Prover:

http://bach.istc.kobe-u.ac.jp/seqprover/

Please use for *checking*, not doing, homework!

Sequent Prover (seqprover)

Sequent: (p->q)/\(q->r) --> p->r [Prove]

Output style: Pretty form

[Top page]

```
Trying to prove with threshold = 0

----------- Ax  ----------- Ax   ----------- Ax  ----------- Ax
p --> q,p,r     p,r --> p,r      p,q --> q,r     p,q,r --> r
--------------------------- L->  --------------------------- L->
      p,q->r --> p,r                   p,q,q->r --> r
      ---------------------------------------------- L->
                     p,p->q,q->r --> r
                     ------------------ R->
                     p->q,q->r --> p->r
                     ----------------------- L/\
                     (p->q)/\(q->r) --> p->r

# Proved in 0 msec.
```

Tableau Prover

http://www.umsu.de/logik/trees/

Please use for *checking*, not doing, homework!