Decidable and Undecidable Theories

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Predicate Calculus is Generally Undecidable

- There is no algorithm that will determine whether or not an arbitrary predicate calculus formula is derivable from a given set of formulas.

- You showed this with your reduction of a Turing machine to Otter. The construction is (or should be) completely general.

Predicate Calculus is Recognizable

- There is an algorithm for **enumerating** all the formulas that can be derived from a given set of formulas.

- In other words, the set of **derivable** formulas is recursively-enumerable, aka **recognizable**.

Predicate Calculus is not Co-Recognizable

- There is no algorithm for determining that a given formula is **not** derivable.

- If there were, then the set of derivable formulas would be decidable, which it isn't, as already mentioned.

Restricted Subsets of the Predicate Calculus

- For example, suppose we require that there be **no predicate symbols**, other than 0-ary ones.

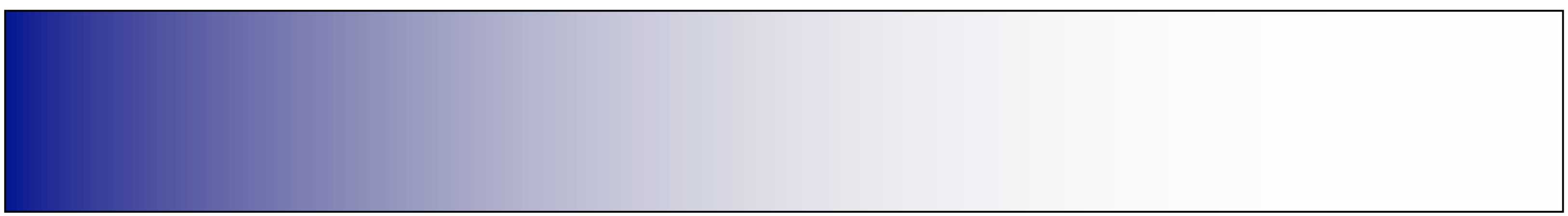
- Then the calculus would effectively turn into propositional calculus, which is decidable.

- This raises the question of whether there are interesting **in-between** cases.

Spectrum of Decidability

Arbitrary predicates, constants, functions

Only 0-ary predicates



undecidable

decidable

Decidable Theories

- However, for certain **special categories of formulas** (cf. W. Ackerman, W. Solvable Cases of the Decision Problem. North-Holland Publishing Co., 1954),

  or for **certain theories**, the question derivability question **can** be decided algorithmically.

- What Makes a Theory Decidable?
  - **Restricting the language** to a specific set of function, predicate, and constant symbols.
  - **Axioms** that constrain the possible **models** to ones where questions can be answered by computation.

What is a Theory?

- A theory restricts the language of predicate calculus to use specific function and predicate symbols
  (e.g. +, x, =, <).

- A theory specifies certain **axioms** and focuses on **theorems**, i.e. the formulas derived from those axioms using the rules of the framework.

Decidability of a Theory

- This is distinct from the decidable of the framework.
- It is generally more restrictive.

- The focus is on whether a particular formula is a theorem or not.

- A theory is **decidable** iff there is an algorithm that decides whether any formula in the language of the theory is a theorem or not.

Completeness of a Theory

- Completeness is related to, but not the same as decidability.

- It is also **not the same** as completeness of a framework. (For example, the predicate calculus *is* complete.)

- **A theory is complete** iff for any closed formula φ in the language of the theory, either φ is a theorem or $\neg\varphi$ is a theorem. [Sometimes this called "negation-complete".]

Completeness of a Framework

- Natural deduction, or other systems equivalent to it, are usually **complete** in the sense that:

  If a formula φ is true for **every model**,
  then φ is derivable.

- However, there can be formulas in the language of a **theory** that are not true in **every** model (even though they **might** be true in models of interest). If this happens, the **theory** is incomplete.

- Hence completeness of a **theory** speaks to **how the well the axioms capture the model(s) of interest.**

More on Theory Completeness

- Why closed formula?
  - Closed formulas (ones with no free variables) evaluate to **true or false** under a given interpretation.
  - Open formulas require an assignment (of values to variables) to give them a truth value.
- Mainly the focus on closed formulas is a convenience, because the truth of a non-closed formula can be equated to the truth of its **closure** (just add an outer $\forall$ for each free variable).

Models, by the way

- Some authors, including Sipser, use the word “model” for “interpretation”.

- Others, including these notes, use “model” only for an interpretation in which the formula is **true**.

Why is completeness good?

- Given an **arbitrary closed formula** in the language of the theory, we’d like to be able to know it is either true or false for all interpretations that satisfy the axioms (i.e. all models).

- That is the main point in having a theory in the first place.

Why is incompleteness not fatal?

- If the theory is incomplete, we just can't know whether an arbitrary formula is true for all interpretations that satisfy the axioms.

- It doesn't mean that there isn't a proof for many true formulas.

- It might be that:
  - We only have ***some*** of the axioms needed to characterize the interpretations of interest.

  - A formula might not be derivable, but its negation might not be either. At the same time, it could be true, or its negation could be true.

Axiomatizability

- A theory is **finitely-axiomatizable** iff all of its theorems can be derived using only a finite set of axioms.
- A theory is **recursively-axiomatizable** iff all of its theorems can be derived using a decidable set of axioms.
- Obviously the latter are of great practical interest. If we can't tell whether something is an axiom, what good is the theory?

Examples

- Group theory is **finitely-**axiomatizable (3 axioms)

- Peano arithmetic is **recursively-** axiomatizable, but not finitely-axiomatizable (due to the induction axiom schema).

Meta-Theorem

- If a theory is both:
  - complete
  - recursively-axiomatizable

  then it is also
  - decidable.

- Proof? Given formula φ, how do we tell whether it is φ or $\neg\varphi$ that is a theorem?

Corollary

- If a theory is both:
  - recursively-axiomatizable
  - undecidable

  then it is also
  - incomplete.

- Peano arithmetic is such a theory.

Decidable does not imply Complete

- For example:
  - ACF: Theory of algebraically-closed fields

Consistency

- A theory is **consistent** iff $\perp$ is not a theorem.
- In other words, a theory is **inconsistent** iff $\perp$ is derivable,
- If we can derive $\perp$ then we can derive **all** formulas, so the theory is worthless: there is no information value in saying something is a theorem.
- Inconsistency is equivalent to there being a φ such that **both** φ and $\neg\varphi$ are derivable.

Trivial Matters

- An inconsistent theory is complete.
- An inconsistent theory is decidable.

Some Known Decidable Theories

- **Theory of Equality**, with no predicate symbols
- **One unary function**, with equality
- **Various limitations on quantifiers**
- **Presburger Arithmetic** (natural numbers with just +, no x, and with a form of induction)
- **Robinson Arithmetic** (natural numbers with +, x, <, but **no induction**)
- **Dense Linear Orders**
- **Real Closed Fields**
- **Boolean Algebras**

Presburger Arithmetic

- $(\forall x)\ \neg(0 = x + 1)$
- $(\forall x)(\forall y)\ \neg(x = y) \rightarrow \neg(x + 1 = y + 1)$
- $(\forall x)\ x + 0 = x$
- $(\forall x)(\forall y)\ (x + y) + 1 = x + (y + 1)$
- If P is any formula involving a single free variable x:

$$(P[0/x] \wedge (\forall x)\ (P \rightarrow P[(x + 1)/x])\ ) \rightarrow (\forall x)\ P$$

- Formulas can be decided based on a **finite automata-like** construction, as shown by M. Presburger in 1929!

- This theory has since been more of interest to computer scientists than logicians: In 1974, Fischer and Rabin proved that any decision procedure requires **at least double exponential time** (2^2^n) where n is the length of the formula. In 1978, Oppen showed a triple exponential upper bound.

From http://en.wikipedia.org/wiki/Presburger_arithmetic

Mojżesz Presburger [1929] proved Presburger arithmetic to be:

- **consistent**
- **complete**: For each statement in Presburger arithmetic, either it is possible to deduce it from the axioms or it is possible to deduce its negation.
- **decidable**: There exists an algorithm which decides whether any given statement in Presburger arithmetic is true or false.

How can a theory for numbers be **complete** if it does not include such things as x (multipy)?

- Although the theory applies to numbers, it can apply to other models as well.
- If it doesn't include x, it only makes statements about the functions it does include.
- In other words, the language of the theory must be taken into account when discussing completeness.

Techniques for Decidability

- often involve some form of **Quantifier-Elimination**:
- Put the formula in **Prenex** form, then come up with some kind of algorithmic analysis based on the matrix and how the quantifiers are stacked.
- A nice example, illustrating the application of computability ideas to logic, follows.

Decidability of arithmetic with only + (no x)

- Based on how **DFAs can check addition** (Sipser prob 1.32):

1.32 Let

$$\Sigma_3 = \left\{ \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \cdots, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}.$$

Σ_3 contains all size 3 columns of 0s and 1s. A string of symbols in Σ_3 gives three rows of 0s and 1s. Consider each row to be a binary number and let

$$B = \{w \in \Sigma_3^* | \text{ the bottom row of } w \text{ is the sum of the top two rows}\}.$$

For example,

LSB first

$$\boxed{\begin{array}{r} 3 \\ +1 \\ \hline 4 \end{array}} \begin{bmatrix}0\\0\\1\end{bmatrix} \begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}1\\1\\0\end{bmatrix} \in B, \qquad \text{but} \qquad \begin{bmatrix}0\\0\\1\end{bmatrix} \begin{bmatrix}1\\0\\1\end{bmatrix} \notin B. \boxed{\begin{array}{r} 1 \\ +0 \\ \hline 3 \text{ not!} \end{array}}$$

Show that B is regular. (Hint: Working with $B^{\mathcal{R}}$ is easier. You may assume the result claimed in Problem 1.31.)

Decidable Theory, continued

Here we use a generalization of this method to present i-tuples of numbers in parallel using an alphabet with 2^i symbols.

We give an algorithm that can determine whether its input, a sentence ϕ in the language of $(\mathcal{N}, +)$, is true in that model. Let

$$\phi = \mathsf{Q}_1 x_1 \, \mathsf{Q}_2 x_2 \, \cdots \, \mathsf{Q}_l x_l \, [\, \psi \,],$$

where $\mathsf{Q}_1, \ldots, \mathsf{Q}_l$ each represent either $\exists$ or $\forall$ and ψ is a formula without quantifiers that has variables $x_1, \ldots, x_l$. For each i from 0 to l, define formula ϕ_i to be

$$\phi_i = \mathsf{Q}_{i+1} x_{i+1} \, \mathsf{Q}_{i+2} x_{i+2} \, \cdots \, \mathsf{Q}_l x_l \, [\, \psi \,].$$

Thus $\phi_0 = \phi$ and $\phi_l = \psi$.

Formula ϕ_i has i free variables. For $a_1, \ldots, a_i \in \mathcal{N}$ write $\phi_i(a_1, \ldots, a_i)$ to be the sentence obtained by substituting the constants $a_1, \ldots, a_i$ for the variables $x_1, \ldots, x_i$ in ϕ_i.

Decidable Theory, continued

For each i from 0 to l, the algorithm constructs a finite automaton A_i that recognizes the collection of strings representing i-tuples of numbers that make ϕ_i true. The algorithm begins by constructing A_l directly, using a generalization of the method in the solution to Problem 1.32. Then, for each i from l down to 1, it uses A_i to construct A_{i-1}. Finally, once the algorithm has A_0, it tests whether A_0 accepts the empty string. If it does, ϕ is true and the algorithm accepts.

PROOF For $i > 0$ define the alphabet

# of rows =
# of free variables
in ϕ_i

$$\Sigma_i = \left\{ \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 1 \end{bmatrix}, \ldots, \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Hence Σ_i contains all size i columns of 0s and 1s. A string over Σ_i represents i binary integers (reading across the rows). We also define $\Sigma_0 = \{[\,]\}$, where $[\,]$ is a symbol.

Decidable Theory, continued

We now present an algorithm that decides $\text{Th}(\mathcal{N}, +)$. On input ϕ where ϕ is a sentence, the algorithm operates as follows. Write ϕ and define ϕ_i for each i from 0 to l, as in the proof idea. For each such i construct a finite automaton A_i from ϕ_i that accepts strings over Σ_i^* corresponding to i-tuples $a_1, \ldots, a_i$ whenever $\phi_i(a_1, \ldots, a_i)$ is true, as follows.

To construct the first machine A_l, observe that $\phi_l = \psi$ is a Boolean combination of atomic formulas. An atomic formula in the language of $\text{Th}(\mathcal{N}, +)$ is a single addition. Finite automata can be constructed to compute any of these individual relations corresponding to a single addition and then combined to give the automaton A_l. Doing so involves the use of the regular language closure constructions for union, intersection, and complementation to compute Boolean combinations of the atomic formulas.

Decidable Theory, continued

Next, we show how to construct A_i from A_{i+1}. $\boxed{\text{If } \phi_i = \exists x_{i+1}\ \phi_{i+1},}$ we construct A_i to operate as A_{i+1} operates, except that it nondeterministically guesses the value of a_{i+1} instead of receiving it as part of the input.

More precisely, A_i contains a state for each A_{i+1} state and a new start state. Every time A_i reads a symbol

$$\begin{bmatrix} b_1 \\ \vdots \\ b_{i-1} \\ b_i \end{bmatrix},$$

where every $b_i \in \{0,1\}$ is a bit of the number a_i, it nondeterministically guesses $z \in \{0,1\}$ and simulates A_{i+1} on the input symbol

$$\begin{bmatrix} b_1 \\ \vdots \\ b_{i-1} \\ b_i \\ z \end{bmatrix}.$$

Decidable Theory, continued

Initially, A_i nondeterministically guesses the leading bits of z corresponding to suppressed leading 0s in b_1 through b_i by nondeterministically branching from its new start state to all states that A_{i+1} could reach from its start state with input strings of the symbols

$$\left\{ \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$$

in Σ_{i+1}. Clearly, A_i accepts its input $(a_1, \ldots, a_i)$ if some a_{i+1} exists where A_{i+1} accepts $(a_1, \ldots, a_{i+1})$.

If $\phi_i = \forall x_{i+1}\ \phi_{i+1}$, it is equivalent to $\neg\exists x_{i+1} \neg\ \phi_{i+1}$. Thus we can construct the finite automaton that recognizes the complement of the language of A_{i+1}, then apply the preceding construction for the $\exists$ quantifier, and finally apply complementation once again to obtain A_i.

Finite automaton A_0 accepts any input iff ϕ_0 is true. So the final step of the algorithm tests whether A_0 accepts ε. If it does, ϕ is true and the algorithm accepts; otherwise, it rejects.

Some Known **Un**decidable Theories

- **Equality and 2 unary function symbols**, otherwise unrestricted
- **Equality and a 2-ary function symbol** , otherwise unrestricted
- **Peano Arithmetic** (natural numbers with +, x, but **with induction**)
- **Rationals**, with +, x, and equality

Three (Meta-)Theorems by Gödel

- Completeness Theorem:
  - Relates to deductive **framework** being complete
- Incompleteness Theorem:
  - Relates to **Number Theory** being incomplete
- Second Incompleteness Theorem:
  - Relates to theories being unable to prove their own consistency

Gödel’s Completeness Theorem

- The completeness of first-order predicate calculus as a **deductive system** was first established by Kurt Gödel in his dissertation in 1929.

- Kurt Gödel, "Über die Vollständigkeit des Logikkalküls", doctoral dissertation, University Of Vienna, 1929.

- Kurt Gödel, "Die Vollständigkeit der Axiome des logischen Funktionen-kalküls", Monatshefte für Mathematik und Physik 37 (1930), 349-360. This article contains the same material as the doctoral dissertation, in a rewritten and shortened form.

Gödel's Incompleteness Theorem
(as refined by Rosser)

- **No consistent recursively-axiomatized extension of number theory is negation-complete.**

- In other words, we can *try* to build up an all-powerful mathematical theory. At a minimum, it must include number theory, which is not asking very much. But such a theory will always be **incomplete**.

- This result, published in 1931, meant that Hilbert's idea of mechanizing all of mathematics could **never** be achieved.

- Kurt Gödel, *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, I.* Monatshefte für Mathematik und Physik, 38 (1931), pp. 173-198. Translated in van Heijenoort: *From Frege to Gödel*. Harvard University Press, 1971., online at http://home.ddc.net/ygg/etext/godel/

- B. Rosser: *Extensions of some theorems of Gödel and Church*. Journal of Symbolic Logic, 1 (1936), N1, pp. 87-91.

Contrapositive

- A complete, consistent theory must be weak, in that in cannot even derive all true formulas about the natural numbers.

Number Theory

- Essentially one with +, *, and induction, usually called **Peano Arithmetic**
- $(\forall x)\ \neg(S(x) = 0)$
- $(\forall x)\ (\forall y)\ ((S(x) = S(y)) \rightarrow (x = y))$
- $(\forall x)\ (x + 0) = x$
- $(\forall x)\ (\forall y)\ (x + S(y)) = S(x+y)$
- $(\forall x)\ (x * 0) = 0$
- $(\forall x)\ (\forall y)\ (\forall z)\ (x*S(y)) = ((x*y) + x)$
- $(\varphi[0/x] \wedge (\forall n)\ (\varphi[n/x] \rightarrow \varphi[S(n)/x])) \rightarrow (\forall n)\varphi[n/x]$

  for **every formula** φ where n is free for x.

How did Gödel’s proof work?

- Similar to the encoding of Turing machine tapes as numbers and Turing machines as partial recursive functions, Gödel first showed how to:
  - encode formulas as numbers
  - encode proofs as numbers

- Such encodings are often called **Gödel numberings**.

How did Gödel’s proof work?

- He then showed how to construct a number-theoretic formula Π (for “provable”), where

  Π(p, f, a) means

  If **f** is the encoding of a formula φ(v) with v representing the one and only free variable, and

- **p** is the encoding of a **proof** of formula φ(a), and

- **a** is an **argument**, substituted for v in φ(v).

- That is, Π(p, f, a) means **p proves φ(a).**

- The actual structure of Π is quite large and complex.

How did Gödel’s proof work?

- Π(p, f, a) means p is a proof of φ(a), where **f encodes** φ.
- Consider the following formula Δ(f) incorporating formula Π:

  Δ(f) is $(\forall p)\ \neg\Pi(p, f, f)$

  meaning there is ***no*** proof of φ(f).
- Let **d** be the encoding of Δ(f). Then **Δ(d)** says

  Δ(d) is ***un***provable.
- **Now *is* Δ(d) provable?** If it is, then by the completeness theorem (for the framework), Δ(d) would then be true. But if it is true, then from the very meaning of Δ(d), it is ***not*** provable, which is a contradiction.
- So Δ(d) is **not provable**.

How did Gödel’s proof work?

- Is the negation, $\neg\Delta(d)$, provable?
- If it were provable, then it would be true (unless the theory is inconsistent), by soundness of the framework.
- But if $\neg\Delta(d)$ is *true*, then $\Delta(d)$ is false, meaning: $\Delta(d)$ ***is*** not not provable, i.e. $\Delta(d)$ is provable, but we just showed that can’t be on the previous page.
- Since neither $\Delta(d)$ nor $\neg\Delta(d)$ is provable, we are forced to conclude that **number theory, *if consistent*, is not (negation-) complete**.
- For a more rigorous proof using the natural deduction framework, see the book by van Dalen, “Logic and Structure”.

Connection with Decidability

- In particular, what does Gödel's incompleteness theorem have to with **decidability** (= solvability)?
- First, the construction of Δ(d) should look roughly similar to the proof of undecidabilty of the acceptance problem **using the recursion theorem**:
  - Δ(d) asserts there is **no** proof of Δ(d)
  - M(x) converges iff M(x) diverges on x.
- Second (and this is not obvious in our brief exposition) the functions used to build up Π, from which Δ is derived, are the **partial recursive functions**.

Decidability of Number Theory

- Number theory is incomplete.
- But it still could conceivably be decidable. We haven't shown that it isn't so far.
- We can, however, reduce A_{TM} to the problem of deciding a formula in number theory.

Number Theory is Undecidable

- A Turing machine computation can be uniformly encoded as a number theoretic functions. In slides on partial-recursive functions, we stated:
  - Halting in i steps on x_0 is expressed by:

$$\mu i\ [P(T(i, x_0)) = 0]$$

- So the logic formula that expresses whether a machine, coded as primitive-recursive functions P and T, halts is:

$$\exists i\ [P(T(i, x_0)) = 0]$$

- This formula is either true or false. If we could determine which, we'd be solving the halting problem.

Gödel’s Second Incompleteness Theorem

- Within any consistent-extension of number theory, there is a formula that expresses the **consistency** of the theory but which is not provable within the theory.

Connections Between Logic, Computability, and Complexity

- A decision problem is in **NP** if it can be solved by a **Non-deterministic Turing machine** in time **Polynomial** in the length of the input.
- A decision problem is in **P** if it can be solved by a **deterministic** Turing machine in Polynomial time.
- A problem is **NP-complete** if it is NP and any other problem in NP **reduces** to it by a polynomial-time reduction.
- The theory of NP came, in part, from Stephen Cook's investigation of complexity of **automatic theorem proving**: The problem of determining whether a **propositional** formula is **satisfiable** is NP-complete.
- It is **unknown** whether P = NP. (It seems **unlikely**, but no one has been able to prove this yet.)
- If P = NP, every NP complete problem (of which there are many) has a polynomial-time algorithm.
- For a list of 88 NP-complete problems, see: http://www.csc.liv.ac.uk/~ped/teachadmin/COMP202/annotated_np.html

Logic and Computing Timeline

- 1822 Babbage: Difference Engine
- 1847 Boole: Laws of Thought
- 1866 Boolean Algebra applied to Switching
- 1890 Hollerith Tabulating Machines
- 1900 Fleming invents Vacuum tube
- 1929 Gödel: Completeness Theorem
- 1931 Gödel: Incompleteness Theorem
- 1935 Church: An unsolvable problem of elementary number theory, lambda calculus
- 1936 Kleene: General recursive functions, unsolvability
- 1936 Post: computing processes (essentially equivalent to Turing machines)
- 1937 Turing invents the Turing machine, uncomputability
- 1939 Stored-program electronic computer
- 1947 First transistor
- 1948 Univac: First commercial electronic computer
- 1954 Fortran
- 1958 Integrated circuit
- 1963 Sutherland's Sketchpad
- 1963 LINC minicomputer
- 1969 Arpanet
- 1971 Microprocessor
- 1972 Pocket calculator
- 1973 Xerox alto, point-and-click
- 1975 Ethernet
- 1976 Apple I
- 1981 IBM PC
- 1983 Visicalc
- 1984 Macintosh