Uncomputability

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Generality of Decision Problems

- Decision problems are about existence of algorithms to test membership in a language that is the subset of all problem encodings.

- Because Turing machines accept languages, it is possible to ask the question about whether a TM exists that does a certain algorithm.

- The fundamental question is whether there is an algorithm. We are not generally interested in answering the question for just one specific case.
Decidability

- A problem is called **decidable** if there is an **algorithm** that will determine a **yes/no** answer for every instance of the problem.

- Equivalently, is there an algorithm that determines membership in the **language** of instances for which the answer is yes.

- The language is said to be a **decidable language**.
There Are Undecidable Languages

• Every decidable language corresponds to a Turing machine that always halts and gives a yes or no answer, per the Church-Turing thesis.

• The set of encodings is countably infinite.

• The set of languages is equivalent to the power set of $\Sigma^*$, which is uncountable.

• So there are languages $L$ for which there is no Turing machine that accepts $L$. 
Meaning of “Enumerate”

• A set is **enumerable** (or “denumerable”) iff it can be put into one-to-one correspondence with the natural numbers or a subset thereof.

• Examples:
  • The set of all **pairs** of natural numbers **is** enumerable.
  • The set of all **subsets** of natural numbers **is not** enumerable.
  • The set of all **finite subsets** of natural numbers **is** enumerable.
Cantor’s Diagonal Argument

- Suppose the set of all languages can be enumerated.

Supposed enumeration of all languages

| L₀ | L₁ | L₂ | L₃ |...
|----|----|----|----|---
| X₀ | X₁ | X₂ | X₃ |...

Have a 1 in row i column j if $x_j \in L_i$. Have a 0 otherwise.
Diagonal Argument

- Have a 1 in row i column j if \( x_j \in L_i \).
  Have a 0 otherwise.

Supposed enumeration of all languages

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Enumeration of \( \Sigma^* \):

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Diagonal Argument

• Where is the flattened flipped diagonal in the enumeration of languages?

Supposed enumeration of all languages

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flattened flipped diagonal
Diagonal Argument

- The flattened flipped diagonal can’t be anywhere in the enumeration. It disagrees with each language in at least one bit.

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Supposed enumeration of all languages

Busted!
Diagonal argument in symbols

- From the supposed enumeration of all languages $L_0, L_1, L_2, ...$ and the known enumeration of $\Sigma^*, x_0, x_1, x_2, ...$, we constructed a new language not in the enumeration after all:
  \[ K = \{ x_i \mid x_i \not\in L_i \} \]

For, if $K$ were in the enumeration it would be $L_k$ for some $k$, thus:
  \[ L_k = \{ x_i \mid x_i \not\in L_i \} \]

which is absurd, because it implies $x_k$ is in $L_k$ iff it isn’t in $L_k$. 
Summary so far, and direction

- The set of all languages cannot be enumerated. There are “too many”.

- The set of all Turing machines *can* be enumerated. Each one can be encoded into a *single element* of $\Sigma^*$ (the machine’s description).

- Thus there must be *some* languages (many, in fact) for which there is no Turing machine.

- That is, *some* languages are not effectively computable.
Can’t we simply add K to the list?

- We could add K to the list, in principle.

- This would yield a new list.

- The diagonal argument can then be repeated on the new list

and so on, until we get tired.
A Specific Undecidable Language

• We know that for each decidable language there is a Turing machine accepting that language.

• Some Turing machines don’t always halt, so we also have some other machines in the mix of all machines.

• If we enumerate all machines, we’ll get all decidable languages, and then some.
A Specific Undecidable Language

- Again use the diagonal argument, only this time, enumerate the Turing machines.

Enumeration of \( \Sigma^* \)

Have a 1 in row \( i \) column \( j \) if \( x_j \in L(T_i) \). Have a 0 otherwise.

Enumeration of all TM’s.
(This does exist.)
All decidable languages are represented.
A Specific Undecidable Language

- Form a new language $K = \{x_i \notin L(T_i)\}$ by “flipping” the diagonal as before. This language is not accepted by any Turing machine, hence is not decidable.

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Enumeration of **all** TM’s.
(This does exist.)
All decidable languages are represented.

*flattened flipped diagonal*
Diagonal Argument in Symbols

- We created language $K = \{ x_i \mid x_i \notin L(T_i) \}$ by “flipping the diagonal”.

- We observed that $K$ is not in our enumeration, i.e. it is not among the enumeration of all TM-acceptable languages.
Another Version, Same Idea

• \(<M>\) is the encoding of machine TM M, for every M.

• Consider language \(K = \{<M> | M \text{ does not accept } <M>\}\).

• There is no Turing machine that accepts \(K\), i.e. \(K\) is undecidable.

• Proof: Suppose some machine, \(N\), does accept \(K\).
• Ask the question whether \(<N> \in K\) or not.
• Then the following are equivalent:
  • \(<N> \in K\) iff, by definition of \(K\)
  • \(N\) does not accept \(<N>\)
  • \(<N> \notin K\) iff, by definition of \(N\) as the machine for \(K\)
• Obviously, we have a contradiction.
Accept vs. Recognize

• A TM accepts (or “decides”) a language iff it always halts, and indicates whether its original input is in the language or not.

• A TM M recognizes a language L iff for each input x:
  • If $x \in L$ then M will halt and indicate acceptance.
  • If $x \notin L$ then M does not indicate acceptance. (M may halt and reject, or it may diverge, i.e. go on forever without halting.)
Notation Refinement

• Let M be a Turing machine.

• By $L(M)$ we will mean the language recognized by M.

• If M always halts, $L(M)$ is also the language accepted by M.
Recognizability vs. Decidability

- A language is (Turing-) **recognizable** iff it is recognized by some TM.

- A language is (Turing-) **decidable** iff it is accepted by some TM.

- Decidable $\rightarrow$ recognizable.

- But is the converse true?
K is not even recognizable, (but its complement is).

- Once again, the language K defined as
  \[ K = \{ <M> \mid <M> \text{ encodes a TM and } <M> \notin L(M) \} \] (1)
is not recognizable by a Turing Machine.
- Suppose K were recognized by some TM, say N, In other words, \( L(N) = K \). \( \quad (2) \)
  Then explore whether \( <N> \in K \).
- \( <N> \in K \) iff \( <N> \notin L(N) \) by the definition of K (1).
  and
- \( <N> \in K \) iff \( <N> \in L(N) \) by the definition of N (2). So our supposition is invalid. K is not recognizable.
- But the **complement** \( K^c \) is recognizable (e.g. using a UTM to recognize \( <M> \in L(M) \)).
Complementarity Theorem

- A language $L$ is decidable iff both it and its complement are recognizable.
- Proof:
  - ($\rightarrow$) If $L$ is decidable, it is recognizable. Furthermore, its complement is decidable by the same machine with accept and reject states interchanged. Hence the complement is decidable.
  - (Continued on next page.)
Complementarity Theorem

- Proof continued:
- \((\leftarrow)\) Suppose \(L\) and its complement are recognizable. Let \(M\) recognize \(L\) and \(N\) recognize its complement. Construct a deterministic TM \(P\) that decides \(L\), as follows: For a given input, \(P\) alternates simulating a step of \(M\) on \(x\) with simulating \(N\) on \(x\). If \(M\) accepts \(x\), then \(P\) accepts \(x\). If \(N\) accepts \(x\), then \(P\) rejects \(x\).

- One of \(M\) or \(N\) will accept \(x\) eventually (by definition of recognition), so \(P\) will decide \(x\). Not both of \(M\) or \(N\) will accept \(x\), because one recognizes the complement of what the other recognizes.
co-recognizability

- A language is called **co-recognizable** iff its complement is recognizable.

- Thus, from the Complementarity Theorem:
  - L decidable iff (L recognizable *and* L co-recognizable)
  - (L recognizable but not co-recognizable) implies L not decidable
  - (L co-recognizable but not recognizable) implies L not decidable

- It is possible for a language to be neither recognizable nor co-recognizable, as we shall see.
More Standard Terminology

- Recall: A language is (Turing-) **decidable** iff it is accepted by some TM.

  Much literature uses the term “**recursive**” as a synonym for “**decidable**”.

- Recall: A language is (Turing-) **recognizable** iff it is recognized by some TM:

  Most literature uses the term “**recursively-enumerable**” (r.e.) *instead of “**recognizable**.”*
Why “recursive”?

- It has to do with the Gödel/Kleene notion of **recursive functions** as being the equivalent of the effectively computable functions.

- A recursive language has a recursive characteristic function.

- Note: Do not read into this anything about the function calling itself, etc. which deals with the way in which the function is **expressed**.
Why “recursively enumerable”?

- It means there is a recursive function that enumerates the language. We’ll soon see that this is equivalent to the language being recognizable by a Turing machine.
Computable Function
(vs. decidable language)

A *function* $f: \Sigma^* \to \Sigma^*$ is *computable* if there is a TM such that if M is started with $x$ on its tape, M will eventually halt with $f(x)$ on its tape.

Computable functions are also called *recursive functions* in the computability literature.
Partial Functions

A *partial function* is like a function, except that it can be undefined for some or all values of arguments.

So it has the *uniqueness* property of a function: \( x = y \) implies \( f(x) = f(y) \), but may lack the *definedness* property, that \( f(x) \) is defined for all \( x \) in the domain.

The *same notation* is usually used for function and partial function, relying on context to resolve the distinction.

Sometimes we write \( f(x) = \perp \) to designate “\( f(x) \) is undefined”. But be aware that \( \perp \) is *no ordinary value*. 
Computable Partial Function

A partial function $f: \Sigma^* \rightarrow \Sigma^*$ is computable if there is a TM such that if $M$ is started with $x$ on its tape, $M$ will eventually halt with $f(x)$ on its tape, or converge (never halt).

Common notation:

- $f(x) \downarrow$ means $f(x)$ is defined.
- $f(x) \uparrow$ means $f(x)$ diverges ($f(x) = \bot$).

In general, there is no computable test for $f(x) = \bot$. It is just a notational convenience.

Computable partial functions are also called partial recursive functions in the literature.
Characteristic Functions

- The **characteristic function** of a language $L \subseteq \Sigma^*$ is a function $ch_L: \Sigma^* \rightarrow \{0, 1\}$ defined by

  $$\forall x \in \Sigma^* \quad ch_L(x) = 1 \text{ if } x \in L$$

  $$0 \text{ otherwise}$$

- Every language has one, and every function of this form determines a language.
Observations

• A language is **decidable** iff it has a computable characteristic function.

• A language is **recognizable** iff it has a computable “partial characteristic function”:
  \[ \forall x \in \Sigma^* \ pch_L(x) = 1 \text{ if } x \in L \]
  undefined otherwise
Enumerability

- The Sipser definition of an Enumerator:

- A Turing machine with a “printer” that prints out all the elements of a language over time.

This could be replaced with a linear “output tape”, or *interleaved* with the squares of the work tape.
The Language Enumerated

- An enumerate **enumerates** a language if, running autonomously, *each* element of the language will get printed at some step ("in finite time").

- If the language enumerated is **infinite**, then the enumerator can never halt.

- If the language enumerated is **finite**, the enumerator may or may not halt (it could print the same element multiple times).
An Alternate Definition of **Enumerator**

- A Turing machine that interprets its input as an encoded natural number \(<i>\), and

- for each value of \(i\), the machine **halts** with an element of \(\Sigma^*\) in a designated area of its tape. This element is declared to be the \(i^{\text{th}}\) element \(x_i\) of a language \(L\).

- The language enumerated is \(L = \{x_0, x_1, x_2, \ldots \}\) the set of strings produced by halting computations.

- (If \(L\) is finite, there will be repetitions in the sequence.)
Why is the Alternate Definition Equivalent to Sipser’s?

• Given a Sipser enumerator $S$, we can construct an Alternate enumerator $A$:
  • With input $<i>$, where $i \geq 0$, $A$ simulates $S$ up until $i+1$ strings have been “printed”. Then $A$ outputs the last string printed as $x_i$.

• Given an alternate enumerator $A$, we can construct a Sipser enumerator $S$: $S$ simulates calls $A$ on different arguments: $A(<0>)$, $A(<1>)$, ..., to produce the elements of the set being enumerated.
Yet Another, More Liberal, Version of **Enumerator**

- A Turing machine that interprets its input as an encoded natural number \(<i>\), and

- for each value of \(i\), the machine, **if it halts**, does so with an element of \(\Sigma^*\) in a designated area of its tape. The \(k^{th}\) element for which the machine halts is declared to be the \(k^{th} \text{ element} \ x_k\) of a language \(L\).

- The language enumerated is \(L = \{x_0, x_1, x_2, \ldots\}\) the set of strings produced by halting computations.

- Since whether the machine halts can’t be predicted, we use “dovetailing” to run multiple computations of the machine at the same time.
Dovetailing

- If S were to call $A(<0>)$, then $A(<1>)$, then $A(<2>)$, ...
  and wait until the previous computation succeeded before going on, there could be a problem: $A(i)$ might diverge.
- To avoid this issue, S simulates:
  - 0 steps of $A(<0>)$,
  - 1 step of $A(<0>)$ then 1 step of $A(<1>)$,
  - ...
  - i steps of $A(<0>)$, $A(<1>)$, ..., $A(<i>)$
  - ...
- Whenever one of the $A(<j>)$ halts, the value on the tape is the next string in the enumeration.
Acceptance Languages

- Sipser’s notation includes languages of the form $A_F$ where $F$ is some formalism, such as DFA, Regular Expressions, Grammars, etc.

- Example: $A_{\text{DFA}}$ is the language of encodings of DFAs with a string the DFA accepts. $\langle B, w \rangle$ is in the language iff:
  - $\langle B \rangle$ encodes a DFA
  - $\langle w \rangle$ encodes an input to the DFA
  - $B$ accepts $w$
- If $\langle B \rangle$ or $\langle w \rangle$ are malformed in some way, then $\langle B, w \rangle$ is simply not in the language.
Examples of Acceptance Languages

• $A_{DFA} = \{<B, w> \mid B \text{ is a DFA accepting } w\}$

• $A_{REX} = \{<R, w> \mid R \text{ is a regular expression and } w \in L(R)\}$.

• $A_{CFG} = \{<G, w> \mid G \text{ is a context-free grammar and } w \in L(G)\}$.

• $A_{TM} = \{<M, w> \mid M \text{ is a Turing machine and } w \in L(M)\}$.

Note: $L(M)$ is the language recognized by $M$. 
Important Note

- One should not infer there is only one specific algorithm for such languages.

- In particular, an algorithm for deciding (or recognizing) membership will not necessarily involve *executing* a particular model.

- Thinking that it does will only lead to trouble.

- Testing whether \( \langle B, w \rangle \in A_{\text{CFG}} \), for example, can be done in a variety of ways (such as?).
Which Languages are Decidable? Recognizable?

- \( A_{DFA} = \{<B, w> \mid B \text{ is a DFA accepting } w\} \)
- \( A_{REX} = \{<R, w> \mid R \text{ is a regular expression and } w \in L(R)\} \).
- \( A_{CFG} = \{<G, w> \mid G \text{ is a context-free grammar and } w \in L(G)\} \).
- \( A_{TM} = \{<M, w> \mid M \text{ is a Turing machine and } w \in L(M)\} \).

Note: \( L(M) \) is the language *recognized* by \( M \).
Why $A_{TM}$ is not decidable.

- Claim: $A_{TM}$ decidable $\rightarrow$ $K$ decidable.

- But we know $K$ is not decidable.

- If $A_{TM}$ were decidable, we could create an algorithm for deciding $K$ as well:
  - With input $<M>$, construct $<M, <M>>$ (the description of $M$ together with its own description) and pass it to the algorithm for $A_{TM}$. $<M, <M>> \notin A_{TM}$ iff $M$ does not accept $<M>$ iff $<M> \in K$.

- Thus an algorithm $A_{TM}$ for can be used to construct an algorithm for $K$, which we previously showed to be impossible.
Corollary: $A_{TM}^c$ (the complement of $A_{TM}$) is not recognizable.

- Why?
The “Halting Problem”

- $\text{HALT}_{\text{TM}} = \{<M, w> \mid M \text{ is a TM and } M \text{ halts on input } w\}$

- $\text{HALT}_{\text{TM}}$ is undecidable.
Proof that $\text{HALT}_{\text{TM}}$ is undecidable

• Suppose $\text{HALT}_{\text{TM}}$ were decidable. That means there is an algorithm for it. We’ll use that algorithm to construct an algorithm for $\text{A}_{\text{TM}}$, which we previously showed to be impossible.

• Algorithm: With input $<M, w>$, first check if $<M, w> \in \text{HALT}_{\text{TM}}$.
  
  • If not, then reject, as M cannot accept w in this case.
  
  • If so, then simulate M on w (e.g. using a universal TM), and accept iff M accepts w.
Summary

• An algorithm for \( \text{HALT}_{\text{TM}} \) could be used to construct an algorithm for \( A_{\text{TM}} \)

thus

• \( \text{HALT}_{\text{TM}} \) decidable \( \rightarrow \) \( A_{\text{TM}} \) decidable

equivalently (contrapositive)

• \( A_{\text{TM}} \) undecidable \( \rightarrow \) \( \text{HALT}_{\text{TM}} \) undecidable
Terminology

• On the previous slide, we have

reduced $A_{TM}$ to the halting problem,

in the sense that if the halting problem were solvable, so would $A_{TM}$ be.

But the latter was already established to be unsolvable.

• Note: It is not correct to say the opposite, that we have reduced the halting problem to $A_{TM}$ (yet, but this can be done).
Notation for Reduction

• If a problem (language) $A$ can be reduced to a problem (language) $B$ we write:

$$A \leq B$$

sort of suggesting that $B$ is “at least as difficult” as $A$.

So if $A$ is unsolvable, so must be $B$. 
Different kinds of reduction

- What we just saw was an example of a **Turing reduction**.

- Language $A \subseteq \Sigma^*$ is **Turing reducible** to $B \subseteq \Sigma^*$, notated $A \leq_T B$ provided:

  an algorithm for deciding $A$ can be implemented by calling an algorithm for deciding $B$ (querying $B$ as if it were an **oracle**). Any number of such calls can be used in general.
Mapping Reducibility $\leq_m$

- Language $A \subseteq \Sigma^*$ is **mapping reducible** to $B \subseteq \Sigma^*$, notated $A \leq_m B$ provided:

  there is some computable $f: \Sigma^* \rightarrow \Sigma^*$ such that

  \[ x \in A \text{ iff } f(x) \in B. \]

- In this case, $f$ is called a **reduction** of $A$ to $B$.

- Mapping reductions are also called “many-to-one” reductions.

- They are a special case of Turing reductions, in which the oracle is only called once, to give the final answer. (Sort of like the **tail-recursive** version of $\leq_T$.)
A $\leq_m$ B and computability

• A $\leq_m$ B implies if B were computable, so would A be:
  
  To determine if x is in A: compute f(x) by machine, determine whether f(x) is in B, to get your answer.

• It also means the contrapositive, that if A is uncomputable, so is B.
Example

- $K \leq_m A_{TM}^c$
- $K = \{<M> | <M> \not\in L(M)\}$
- $A_{TM}^c = \{<M, w> | w \not\in L(M)\}$
- The reduction mapping in this case is:
  \[ f(<M>) = <M, M> \]
\leq_m \text{ Mapping need not be 1-1}
How about $\text{HALT}_\text{TM} \leq A_{\text{TM}}$?

- Here we can use a mapping reduction.
- Assume $A_{\text{TM}}$ is decidable. Map $\text{HALT}_\text{TM}$ into it.

- Given a machine and input $<M, w>$, map it to $<M', w>$ by creating $<M'>$ from $<M>$:
  - Change all \text{halting} (accepting or rejecting) states of $M$ into \text{accepting} states of $M'$, leaving everything else unchanged.
  - The mapping reduction $f$ does this transformation:
    \[ f(<M, w>) = <M', w> . \]
  - So $M$ halts on $w$ iff $M'$ accepts $w$.
  - i.e. $<M, w> \in \text{HALT}_\text{TM}$ iff $<M', w> \in A_{\text{TM}}$. 
Emptiness Problems

- $E_{\text{DFA}} = \{ <B> \mid B \text{ is a DFA accepting no strings} \}$

- $E_{\text{REX}} = \{ <R> \mid R \text{ is a regular expression and } L(R) = \emptyset \}$

- $E_{\text{CFG}} = \{ <G> \mid G \text{ is a context-free grammar and } L(G) = \emptyset \}$.

- $E_{\text{CSG}} = \{ <G> \mid G \text{ is a context-sensitive grammar and } L(G) = \emptyset \}$.

- $E_{\text{TM}} = \{ <M> \mid M \text{ is a Turing machine and } L(M) = \emptyset \}$

- Which of these are decidable? (Not necessarily obvious).
$E_{TM}$ is not decidable

- Use a mapping reduction $A_{TM} \leq_m E_{TM}$, the latter accepting those $<M>$ where $L(M)$ is non-empty, in other words $M$ accepts some input.

- Transformation: With input $<M, w>$, construct machine $<M'>$ such that:
  - $<M'> \in E_{TM}$
  - With input $x$, erase $x$, write $w$ on the tape, then behave as $M$. 

- $M'$: With input $x$, erase $x$, write $w$ on the tape, then behave as $M$. 

Property of $M'$

- $M'$ does exactly the same thing on every input $x$.
  - It accepts $x$ iff $M$ accepts $w$.
  - It rejects $x$ iff $M$ rejects $w$.
  - It diverges on $x$ iff $M$ diverges on $w$.

- So it accepts some input iff $M$ accepts $w$. 
Is either $E^T_M$ or its complement recognizable?
A \leq_m B and recognizability

- A \leq_m B also implies that if B were recognizable, so would A be.

- To recognize if x is in A: compute f(x) by machine, ask whether f(x) is in B.
  
  If the answer is affirmative, then the original x was in A.
  
  If we never get an answer, then we don’t get answer for whether the original x was in A either.

- implies if A is not recognizable, neither is B.
A \leq_m B and co-recognizability

• A \leq_m B similarly implies that if B were co-recognizable, so would A be.

• A \leq_m B implies if A is not co-recognizable, neither is B.
\[ A^c \leq_m B \]

- \( A^c \leq_m B \) means the complement of \( A \) reduces to \( B \). This is equivalent to \( A \leq_m B^c \).
Sipser Problem 5.7

- If $A$ is recognizable, and $A \leq_m A^c$, then $A$ is decidable.

- Proof: If $A \leq_m A^c$, then also $A^c \leq_m A$, per the previous slide. But since $A$ is recognizable, by $A^c \leq_m A$, $A^c$ is also recognizable. But then both $A$ and are $A^c$ recognizable, so $A$ is decidable from the complementarity lemma.
Example

- We already showed $A_{TM} \leq_m E_{c_{TM}}$.
- In effect, the same mapping gives $A_{c_{TM}} \leq_m E_{TM}$.
- But $A_{c_{TM}}$ is not recognizable.
- Therefore $E_{TM}$ is not recognizable.
- However, $E_{c_{TM}}$ is co-recognizable.
“ALL” Problems

- \( \text{ALL}_{\text{DFA}} = \{ <B> | B \text{ is a DFA accepting all strings} \} \)
  ("all" means all of \( \Sigma^* \) where \( \Sigma \) is the alphabet of the DFA)

- \( \text{ALL}_{\text{REX}} = \{ <R> | R \text{ is a regular expression and } L(R) = \Sigma^* \} \)

- \( \text{ALL}_{\text{CFG}} = \{ <G> | G \text{ is a context-free grammar and } L(G) = \Sigma^* \} \).

- \( \text{ALL}_{\text{CSG}} = \{ <G> | G \text{ is a context-sensitive grammar and } L(G) = \Sigma^* \} \).

- \( \text{ALL}_{\text{TM}} = \{ <M> | M \text{ is a Turing machine and } L(M) = \Sigma^* \} \)

Which of these are decidable? (Not necessarily obvious).
$\text{ALL}_{\text{TM}}$

- Is $\text{ALL}_{\text{TM}}$ decidable?
- What is your intuition?
- How would you prove it?
- (Remember the tune “All or Nothing At All”.)
- What about recognizable or co-recognizable, if not decidable.
Fixed-String Acceptance

- Suppose $x_0$ is a fixed string, such as the empty string (blank tape).

- Is there an algorithm for determining whether an arbitrary TM accepts $x_0$?

- We call the corresponding language acceptance problem $\text{Accepts-}x_0^{\text{TM}}$. 
Accepts-$x_0 \text{ TM}$ is unsolvable, for any fixed $x_0$.

- See if you can reduce one of the other unsolvable problems to this problem.
Regular_{TM} is Undecidable

- Define \(<M>\in\text{Regular}_{TM}\) iff \(L(M)\) is regular.

- Reduce a known unsolvable problem to \(\text{Regular}_{TM}\).

- Try a mapping reduction.
Reduction to Regular$_{\text{TM}}$

- Reduce $A_{\text{TM}}$ to Regular$_{\text{TM}}$ as follows:

- Given an instance $<M, w>$, create $<M'>$ as follows:
  - $M'$ with input $x$: check whether $x$ has the form $0^n1^n$ for some $n$.
  - If $x$ does not have the form, reject.
  - If $x$ does have the form, run $M$ on $w$, accepting when and if $M$ accepts $w$. (This may, of course, diverge, in which case $M$ cannot accept $w$.)
Reduction to Regular<sub>TM</sub>

- Thus M’ accepts exactly strings of the form 0<sup>n</sup>1<sup>n</sup> iff M accepts w. Otherwise M accepts ∅.

- So L(M’) is either
  - regular (∅) or
  - non-regular (\{0<sup>n</sup>1<sup>n</sup> | n ≥ 0\})
depending on whether M accepts w.

[We’ve thus shown A<sub>TM</sub> ≤<sub>m</sub> Regular<sup>c</sup> TM.]

What about Regular itself?
Functional vs. Structural Properties

- We have been using TM’s to represent languages.

- When a property of a machine depends only on the language and not the specific TM used to represent the language, we call this a functional property.

- When a property depends on the specific details of a machine, it is called a structural property.
Functional vs. Structural Examples

- **Functional:**
  - $L(M) = \emptyset$
  - $L(M)$ is regular
  - $L(M)$ is infinite
  - $L(M)$ is decidable
  - $L(M)$ is recognizable (always true, by definition)

- **Structural:**
  - $M$ has more than 100 reachable control states
  - $M$ writes a non-blank character on its tape when started on an empty tape.
  - $M$ reverses its direction of head travel at least once on each input.
Decidable vs. Undecidable Structural Properties

- M has more than 100 control states.
- M reaches a specific control state when started on an empty tape.
- M writes a non-blank character on its tape when started on an empty tape.
- M uses more than a specified amount of tape when started on a certain input.

Which of these properties is decidable?
Rice’s Theorem, Informally

- Practically **no** functional properties are decidable for TM recognizable languages.

- The caveat here is that the theorem only applies to functional properties, not structure ones.
**Trivial Functional Properties**

- A functional property of a recognizable language is called *trivial* if it is either:
  - true for *no* recognizable language, or
  - true for *all* recognizable languages

- A *non-trivial property*, then, holds for *some* recognizable languages, but *not all*. 
Rice’s Theorem

• Any non-trivial functional property of the recognizable languages is not decidable.

• Put another way:

For any non-trivial functional property, there is no algorithm that will determine whether or not the language recognized by a given TM has the property.
Observations about Non-Trivial Properties P

- Any given language has property P, or it does not.
- A language L has property P iff L does not have \( \neg P \).
- To decide P, it is adequate to decide the complementary property \( \neg P \), since yes/no answers are required in both cases.
Proof of Rice’s Theorem (1 of 3)

- Suppose $P$ is a non-trivial property.

- **Critical assumption:** The empty language $\emptyset$ does not have property $P$. If the opposite is true, then interchange $P$ and $\neg P$ so that the assumption is true. [This would be necessary in the case of $P = \text{Regular}$, for example.]

- Let $L_P$ be some arbitrary recognizable language with property $P$ (which must exist, because $P$ is non-trivial). We know that $L_P$ is distinct from $\emptyset$, by the assumption above. Let $M_P$ be a machine recognizing the chosen language $L_P$.

- **Plan:** Reduce the acceptance problem to that of deciding property $P$, which will imply that there is no algorithm for the latter. This will hinge on having the $P$ decider differentiate between $L_P$, which has property $P$, and $\emptyset$, which does not.
Proof of Rice’s Theorem (2 of 3)

• The reduction works as follows: Suppose we have an algorithm for testing property P for any input \( <M> \).

• We can then test whether an arbitrary Turing machine \( M \) accepts an arbitrary input \( w \) by constructing a machine \( M' \) with the specifications on the following page.
Proof of Rice’s Theorem (3 of 3)

- **M’**: with input $x$, (temporarily set aside $x$ and) start **behaving as** $M$ on $w$.

- **If** $M$ on $w$ **accepts**, then continue by behaving as $M_p$ on the original input $x$.

- Thus if and when $M$ accepts $w$, $M'$ will accept $x$ iff $M_p$ accepts $x$, i.e. $L(M') = L(M_p)$. So in this case, $L(M')$ **has** property P, because $M_p$ was selected to have it.

- **If** $M$ on $w$ **does not terminate**, then $L(M') = \emptyset$, which by the critical assumption, **does not** have property P.

- So if we can test an arbitrary machine whether its language has property P, we can test $M'$ in particular. But the answer to this test determines whether or not $M$ accepts $w$. 
Diagram of $M'$ as constructed
(dashed lines = control, solid = data)

$M'$

$X$  
(arg to $M'$)

$<M, w>$  
(not arguments to $M'$ but used in its construction)

start here  
run $M$ on $w$

$L_P$  
Accept/Reject

Accept/Reject

M accepts $w$

M rejects $w$ or diverges
Note

• Although P was discussed as a property of language recognized by a Turing machine, the same proof works in the case that:

  P is a property of the partial function computed by a Turing machine.

• This means, for example, there is no algorithm that will decide equivalence of an arbitrary machine’s partial function to that of a given machine.
Languages neither recognizable nor co-recognizable.

• $E_{\text{TM}}$ is neither.

• Proof that $E_{\text{TM}}$ is not recognizable:
  Use reduction from $A_{\text{TM}}$ (which is not co-recognizable) to $E_{\text{TM}}^c$. Given $<M, w>$, construct $M_1$ and $M_2$ thus:
  • $M_1$ always rejects.
  • $M_2$ on any input, behaves as $M$ on $w$.

• So $<M_1, M_2> \notin E_{\text{TM}}$ iff $<M, w> \in A_{\text{TM}}$. 
Languages neither recognizable nor co-recognizable.

- $\text{EQ}_{\text{TM}}$ is neither.

- Proof that $\text{EQ}^c_{\text{TM}}$ is not recognizable:
  Use reduction from $A_{\text{TM}}$ (which is not co-recognizable) to $\text{EQ}_{\text{TM}}$. Given $<M, w>$, construct $M_1$ and $M_2$ thus:
  - $M_1$ always accepts.
  - $M_2$ on any input, behaves as $M$ on $w$.
- So $<M_1, M_2> \in \text{EQ}_{\text{TM}}$ iff $<M, w> \in A_{\text{TM}}$. 
Languages neither recognizable nor co-recognizable.

- $\text{ALL}_{\text{TM}}$ is neither.
- To show $\text{ALL}_{\text{TM}}$ is not co-recognizable, give a mapping reduction from $\text{A}_{\text{TM}}$ (which is not co-recognizable), to $\text{ALL}_{\text{TM}}$.
  
  $\text{f}(<M, w>) = M'$, where $M'(x) = M(w)$. Thus $M'$ accepts all iff $M$ accepts $w$.

- But how to show $\text{ALL}_{\text{TM}}$ is not recognizable??
ALL$_{TM}$ is not recognizable

- We prove this by $\text{HALT}_{TM}^c \leq_m \text{ALL}_{TM}$.
- Assume that $\text{ALL}_{TM}$ is recognizable.
- Let $<M, w>$ be an arbitrary machine with input. The reduction mapping constructs $M'$ to behave as follows:
  - $M'(x)$ simulates $M(w)$ for $|x|$ steps.
  - If $M(w)$ fails to halt within $|x|$ steps, accept $x$.
  - Otherwise reject $x$.
- Hence $M'$ accepts all inputs iff $M(w)$ does not halt.