Uncomputability

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April 2010

Generality of Decision Problems

- Decision problems are about existence of algorithms to test membership in a language that is the subset of all problem encodings.

- Because Turing machines accept languages, it is possible to ask the question about whether a TM exists that does a certain **algorithm**.

- The fundamental question is whether there is an **algorithm**. We are not generally interested in answering the question for just one specific case.

Decidability

- A problem is called **decidable** if there is an **algorithm** that will determine a **yes/no** answer for every instance of the problem.
- Equivalently, is there an algorithm that determines membership in the **language** of instances for which the answer is yes.
- The language is said to be a **decidable language**.

There Are Undecidable Languages

- Every decidable language corresponds to a Turing machine that always halts and gives a yes or no answer, per the Church-Turing thesis.

- The set of **encodings** is countably infinite.

- The set of **languages** is equivalent to the power set of Σ^*, which is **uncountable**.

- So there are languages L for which there is no Turing machine that accepts L.

Meaning of “Enumerate”

- A set is **enumerable** (or “denumerable”) iff it can be put into one-to-one correspondence with the natural numbers or a subset thereof.

- Examples:
  - The set of all **pairs** of natural numbers **is** enumerable.
  - The set of all **subsets** of natural numbers **is not** enumerable.
  - The set of all **finite subsets** of natural numbers **is** enumerable.

Cantor's Diagonal Argument

- Suppose the set of all languages can be enumerated.

Enumeration of Σ^*

$x_0 \quad x_1 \quad x_2 \quad x_3 \ . \ . \ .$

L_0

L_1

L_2

L_3

.
.
.

Supposed enumeration of ***all*** languages

Have a 1 in row i column j if $x_j \in L_i$. Have a 0 otherwise.

Diagonal Argument

- Have a 1 in row i column j if $x_j \in L_i$. Have a 0 otherwise.

Enumeration of Σ^*

Supposed enumeration of ***all*** languages

$$
\begin{array}{c|cccccccc}
 & x_0 & x_1 & x_2 & x_3 & . & . & . \\
\hline
L_0 & 0 & 1 & 0 & 1 & 0 & 1 & \\
L_1 & 1 & 0 & 1 & 0 & 1 & 0 & \\
L_2 & 0 & 0 & 1 & 1 & 0 & 1 & \ldots \\
L_3 & & & & & & & \\
\vdots & & & & & & &
\end{array}
$$

Diagonal Argument

- Where is the flattened flipped diagonal in the enumeration of languages?

Enumeration of Σ^*

Supposed enumeration of ***all*** languages

	x_0	x_1	x_2	x_3	. . .		
L_0	0	1	0	1	0	1	
L_1	1	0	1	0	1	0	
L_2	0	0	1	1	0	1	...
L_3							
⋮							

flattened flipped diagonal

1 1 0 ...

Diagonal Argument

- The flattened flipped diagonal can’t be anywhere in the enumeration. It disagrees with each language in at least one bit.

Supposed enumeration of ***all*** languages

Busted!

	x_0	x_1	x_2	x_3	. . .		
L_0	0	1	0	1	0	1	
L_1	1	0	1	0	1	0	
L_2	0	0	1	1	0	1	…
L_3							
⋮							

flattened flipped diagonal

1 1 0 …

Diagonal argument in symbols

- From the supposed enumeration of all languages L_0, L_1, L_2, … and the known enumeration of Σ^*, x_0, x_1, x_2, …, we constructed a new language not in the enumeration after all:

  $$K = \{x_i \mid x_i \notin L_i\}$$

  For, if K were in the enumeration it would be L_k for some k, thus:

  $$L_k = \{x_i \mid x_i \notin L_i\}$$

  which is absurd,
  because it implies x_k is in L_k iff it isn't in L_k.

Summary so far, and direction

- The set of all languages cannot be enumerated. There are “too many”.
- The set of all Turing machines *can* be enumerated. Each one can be encoded into a *single element* of Σ^* (the machine’s description).
- Thus there must be *some* languages (many, in fact) for which there is no Turing machine.
- That is, *some* languages are not effectively computable.

Can’t we simply add K to the list?

- We could add K to the list, in principle.
- This would yield a new list.
- The diagonal argument can then be repeated on the new list

and so on, until we get tired.

A Specific Undecidable Language

- We know that for each decidable language there is a Turing machine accepting that language.
- Some Turing machines don’t always halt, so we also have some other machines in the mix of all machines.
- If we enumerate all machines, we’ll get all decidable languages, and then some.

A Specific Undecidable Language

- Again use the diagonal argument, only this time, **enumerate the Turing machines.**

Enumeration of Σ^*

$x_0 \quad x_1 \quad x_2 \quad x_3 \; . \; . \; .$

T_0

T_1

T_2

T_3

.
.
.

Enumeration of **all** TM's.
(This does exist.)
All decidable languages are represented.

Have a 1 in row i column j if $x_j \in L(T_i)$. Have a 0 otherwise.

A Specific Undecidable Language

- Form a new language $K = \{x_i \notin L(T_i)\}$ by "flipping" the diagonal as before. This language is not accepted by any Turing machine, hence is not decidable.

Enumeration of **all** TM's.
(This does exist.)
All decidable languages are represented.

	x_0	x_1	x_2	x_3	. . .		
T_0	0	1	0	1	0	1	
T_1	1	0	1	0	1	0	
T_2	0	0	1	1	0	1	...
T_3							
⋮							

flattened flipped diagonal

1 1 0 ...

Diagonal Argument in Symbols

- We created language $K = \{x_i \mid x_i \notin L(T_i)\}$ by "flipping the diagonal".

- We observed that K is **not in our enumeration**, i.e. it is **not** among the enumeration of all TM-acceptable languages.

Another Version, Same Idea

- <M> is the encoding of machine TM M, for every M.

- Consider language K =
  {<M> | M does not accept <M>}.

- **There is no Turing machine that accepts K, i.e. K is undecidable.**

- Proof: **Suppose some machine, N, does accept K**.
- Ask the question whether <N> ∈ K or not.
- Then the following are equivalent:
  - <N> ∈ K — iff, by definition of K
  - N does not accept <N>
  - <N> ∉ K — iff, by definition of N as the machine for K
- Obviously, we have a **contradiction**.

Accept vs. Recognize

- A TM **accepts** (or "**decides**") a language iff it always halts, and indicates whether its original input is in the language or not.

- A TM M **recognizes** a language L iff for each input x:
  - If $x \in L$ then M will halt and indicate acceptance.
  - If $x \notin L$ then M does **not** indicate acceptance.
    (M **may halt** and reject, or it **may diverge**, i.e. go on forever without halting.)

Notation Refinement

- Let M be a Turing machine.
- By L(M) we will mean the language **recognized** by M.
- **If** M always halts, L(M) is **also** the language **accepted** by M.

Recognizability vs. Decidability

- A language is (Turing-) **recognizable** iff it is recognized by some TM.
- A language is (Turing-) **decidable** iff it is accepted by some TM.
- Decidable $\rightarrow$ recognizable.
- But is the converse true?

K is not even recognizable, (but its complement is).

- Once again, the language K defined as
  K = {<M> | <M> encodes a TM and <M> $\notin$ L(M)}(1)
  is not recognizable by a Turing Machine.
- Suppose K were recognized by some TM, say N,
  In other words, L(N) = K. (2)

Then explore whether <N> $\in$ K.

- <N> $\in$ K iff <N> $\notin$ L(N) by the definition of K (1).
  and
- <N> $\in$ K iff <N> $\in$ L(N) by the definition of N (2).
  So our supposition is invalid. K is not recognizable.
- But the **complement** K^c is recognizable
  (e.g. using a UTM to recognize <M> $\in$ L(M)).

Complementarity Theorem

- A language L is decidable iff both it and its complement are recognizable.
- Proof:
- ($\rightarrow$) If L is decidable, it is recognizable. Furthermore, its complement is decidable by the same machine with accept and reject states interchanged. Hence the complement is decidable.
- (Continued on next page.)

Complementarity Theorem

- Proof continued:
- (←) Suppose L and its complement are recognizable. Let M recognize L and N recognize its complement. Construct a deterministic TM P that decides L, as follows: For a given input, P alternates simulating a step of M on x with simulating N on x. If M accepts x, then P accepts x. If N accepts x, then P rejects x.

- One of M or N will accept x eventually (by definition of recognition), so P will decide x. Not both of M or N will accept x, because one recognizes the complement of what the other recognizes.

co-recognizability

- A language is called **co-recognizable** iff its complement is recognizable.

- Thus, from the Complementarity Theorem:
  - L decidable iff (L recognizable *and* L co-recognizable)
  - (L recognizable but not co-recognizable) implies L not decidable
  - (L co-recognizable but not recognizable) implies L not decidable

- It is possible for a language to be neither recognizable nor co-recognizable, as we shall see.

More Standard Terminology

- Recall: A language is (Turing-) **decidable** iff it is accepted by some TM.

  Much literature uses the term
  “recursive” *as a synonym for* **“decidable”.**

- Recall: A language is (Turing-) **recognizable** iff it is recognized by some TM:

  Most literature uses the term
  “recursively-enumerable” (r.e.)
  instead of **“recognizable.**

Why “recursive”?

- It has to do with the Gödel/Kleene notion of **recursive functions** as being the equivalent of the effectively computable functions.
- A recursive language has a recursive characteristic function.
- Note: Do not read into this anything about the function calling itself, etc. which deals with the way in which the function is *expressed*.

Why “recursively enumerable”?

- It means there is a recursive function that **enumerates** the language. We’ll soon see that this is equivalent to the language being recognizable by a Turing machine.

Computable Function
(vs. decidable language)

A ***function*** f: $\Sigma^* \rightarrow \Sigma^*$ is ***computable*** if there is a TM such that if M is started with x on its tape, M will eventually halt with f(x) on its tape.

Computable functions are also called **recursive functions** in the computability literature.

Partial Functions

A ***partial function*** is like a function, except that it can be undefined for some or all values of arguments.

So it has the **uniqueness** property of a function: $x = y$ implies $f(x) = f(y)$, but may lack the **definedness** property, that $f(x)$ is defined for all x in the domain.

The **same notation** is usually used for function and partial function, relying on context to resolve the distinction.

Sometimes we write $f(x) = \perp$ to designate "$f(x)$ is undefined". But be aware that $\perp$ **is no ordinary value**.

Computable Partial Function

A ***partial function*** f: $\Sigma^* \to \Sigma^*$ is ***computable*** if there is a TM such that if M is started with x on its tape, M will **eventually halt** with f(x) on its tape,
or **diverge** (never halt).

Common notation:

$f(x)\downarrow$ means f(x) is defined.

$f(x)\uparrow$ means f(x) diverges ($f(x) = \perp$).

In general, there is no computable test for $f(x) = \perp$.
It is just a notational convenience.

Computable partial functions are also called
partial recursive functions in the literature.

Characteristic Functions

- The **characteristic function** of a language $L \subseteq \Sigma^*$ is a function $ch_L: \Sigma^* \to \{0, 1\}$ defined by

  $$\forall x \in \Sigma^* \qquad ch_L(x) = \begin{cases} 1 & \text{if } x \in L \\ 0 & \text{otherwise} \end{cases}$$

- Every language has one, and every function of this form determines a language.

Observations

- A language is **decidable** iff it has a computable characteristic function.

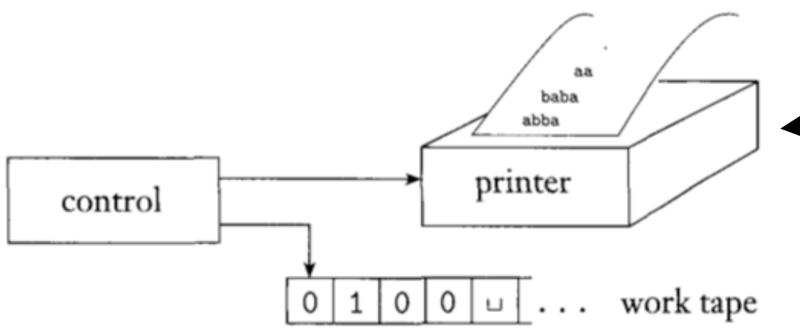
- A language is **recognizable** iff it has a computable "partial characteristic function":

$$\forall x \in \Sigma^* \; pch_L(x) = \begin{cases} 1 & \text{if } x \in L \\ \text{undefined} & \text{otherwise} \end{cases}$$

Enumerability

- The Sipser definition of an Enumerator:

- A Turing machine with a “printer” that prints out all the elements of a language over time.



This could be replaced with a linear “output tape”, or *interleaved* with the squares of the work tape.

The Language Enumerated

- An enumerate **enumerates** a language if, running autonomously, *each* element of the language will get printed at some step ("in finite time").

- If the language enumerated is **infinite**, then the enumerator can never halt.

- If the language enumerated is **finite**, the enumerator may or may not halt (it could print the same element multiple times).

An Alternate Definition of **Enumerator**

- A Turing machine that interprets its input as an encoded natural number <i>, and

- for each value of i, the machine **halts** with an element of Σ^* in a designated area of its tape. This element is declared to be the **i^{th} element** x_i of a language L.

- The language enumerated is $L = \{x_0, x_1, x_2, \ldots\}$ the set of strings produced by halting computations.

- (If L is finite, there will be repetitions in the sequence.)

Why is the Alternate Definition Equivalent to Sipser's?

- Given a Sipser enumerator S, we can construct an Alternate enumerator A:
  - With input <i>, where $i \geq 0$, A simulates S up until i+1 strings have been "printed". Then A outputs the last string printed as x_i.

- Given an alternate enumerator A, we can construct a Sipser enumerator S: S simulates calls A on different arguments: A(<0>), A(<1>), …, to produce the elements of the set being enumerated.

Yet Another, More Liberal, Version of **Enumerator**

- A Turing machine that interprets its input as an encoded natural number <i>, and

- for each value of i, the machine, **if it halts**, does so with an element of Σ^* in a designated area of its tape.
  The k^{th} element for which the machine halts is declared to be the **k^{th} element** x_k of a language L.

- The language enumerated is $L = \{x_0, x_1, x_2, \ldots\}$ the set of strings produced by halting computations.

- Since whether the machine halts can't be predicted, we use "dovetailing" to run multiple computations of the machine at the same time.

Dovetailing

- If S were to call A(<0>), then A(<1>), then A(<2>), … and wait until the previous computation succeeded before going on, there could be a problem: A(i) might **diverge**.
- To avoid this issue, S simulates:
  - 0 steps of A(<0>),
  - 1 step of A(<0>) then 1 step of A(<1>),

    . . .
  - i steps of A(<0>), A(<1>), …., A(<i>)

    . . .
- Whenever one of the A(<j>) halts, the value on the tape is the next string in the enumeration.

Acceptance Languages

- Sipser's notation includes languages of the form A_F where F is some formalism, such as DFA, Regular Expressions, Grammars, etc.

- Example: A_{DFA} is the language of encodings of DFAs with a string the DFA accepts.
  <B, w> is in the language iff:
  - <B> encodes a DFA
  - <w> encodes an input to the DFA
  - B accepts w
- If <B> or <w> are malformed in some way, then <B, w> is simply not in the language.

Examples of Acceptance Languages

- $A_{DFA} = \{\langle B, w\rangle \mid B$ is a DFA accepting $w\}$
- $A_{REX} = \{\langle R, w\rangle \mid R$ is a regular expression and $w \in L(R)\}$.
- $A_{CFG} = \{\langle G, w\rangle \mid G$ is a context-free grammar and $w \in L(G)\}$.
- $A_{TM} = \{\langle M, w\rangle \mid M$ is a Turing machine and $w \in L(M)\}$.

Note: $L(M)$ is the language *recognized* by M.

Important Note

- One should not infer there is only one specific algorithm for such languages.
- In particular, an algorithm for deciding (or recognizing) membership will not necessarily involve **executing** a particular model.
- Thinking that it does will only lead to trouble.
- Testing whether $\langle B, w \rangle \in A_{CFG}$, for example, can be done in a variety of ways (such as?).

Which Languages are Decidable? Recognizable?

- A_{DFA} = {<B, w> | B is a DFA accepting w}
- A_{REX} = {<R, w> | R is a regular expression and w $\in$ L(R)}.
- A_{CFG} = {<G, w> | G is a context-free grammar and w $\in$ L(G)}.
- A_{TM} = {<M, w> | M is a Turing machine and w $\in$ L(M)}.

Note: L(M) is the language *recognized* by M.

Why A_{TM} is not decidable.

- Claim: A_{TM} decidable $\rightarrow$ K decidable.

- But we know K is not decidable.

- If A_{TM} were decidable, we could create an algorithm for deciding K as well:
  - With input <M>, construct <M, <M>> (the description of M together with its own description) and pass it to the algorithm for A_{TM}. <M, <M>> $\notin$ A_{TM} iff M does not accept <M> iff <M> $\in$ K.

- Thus an algorithm A_{TM} for can be used to construct an algorithm for K, which we previously showed to be impossible.

Corollary: A^c_{TM} (the complement of A_{TM}) is not recognizable.

- Why?

The "Halting Problem"

- $HALT_{TM}$ =
  {<M, w> | M is a TM and M halts on input w}

- $HALT_{TM}$ is undecidable.

Proof that $HALT_{TM}$ is undecidable

- Suppose $HALT_{TM}$ were decidable. That means there is an algorithm for it. We'll use that algorithm to **construct an algorithm for $\mathbf{A_{TM}}$**, which we previously showed to be impossible.

- Algorithm: With input <M, w>, first check if <M, w> $\in HALT_{TM}$.
  - **If not**, then **reject**, as M cannot accept w in this case.
  - **If so**, then simulate M on w (e.g. using a universal TM), and accept iff M accepts w.

Summary

- An algorithm for $HALT_{TM}$ could be used to construct an algorithm for A_{TM}

  thus

- $HALT_{TM}$ decidable $\rightarrow$ A_{TM} decidable

  equivalently (contrapositive)

- A_{TM} **un**decidable $\rightarrow$ $HALT_{TM}$ **un**decidable

Terminology

- On the previous slide, we have

  reduced $\mathbf{A_{TM}}$ to the **halting problem**,

  in the sense that **if** the halting problem were solvable, so would $\mathbf{A_{TM}}$ be.

  But the latter was already established to be unsolvable.

- Note: It is **not correct** to say the opposite, that we have reduced the halting problem to $\mathbf{A_{TM}}$ (yet, but this can be done).

Notation for Reduction

- If a problem (language) A can be reduced to a problem (language) B we write:

  $$A \leq B$$

  sort of suggesting that B is "at least as difficult" as A.

  So if A is unsolvable, so must be B.

Different kinds of reduction

- What we just saw was an example of a **Turing reduction**.

- Language $A \subseteq \Sigma^*$ is ***Turing reducible*** to $B \subseteq \Sigma^*$, notated $A \leq_T B$ provided:

  an algorithm for deciding A can be implemented by calling an algorithm for deciding B (querying B as if it were an ***oracle***). Any number of such calls can be used in general.

Mapping Reducibility $\leq_m$

- Language $A \subseteq \Sigma^*$ is ***mapping reducible*** to $B \subseteq \Sigma^*$, notated $A \leq_m B$ provided:

  there is some computable $f: \Sigma^* \rightarrow \Sigma^*$ such that

  $x \in A$ iff $f(x) \in B$.

- In this case, f is called a **reduction** of A to B.
- Mapping reductions are also called "many-to-one" reductions.
- They are a special case of Turing reductions, in which the oracle is only called once, to give the final answer. (Sort of like the **tail-recursive** version of $\leq_T$.)

$A \leq_m B$ and computability

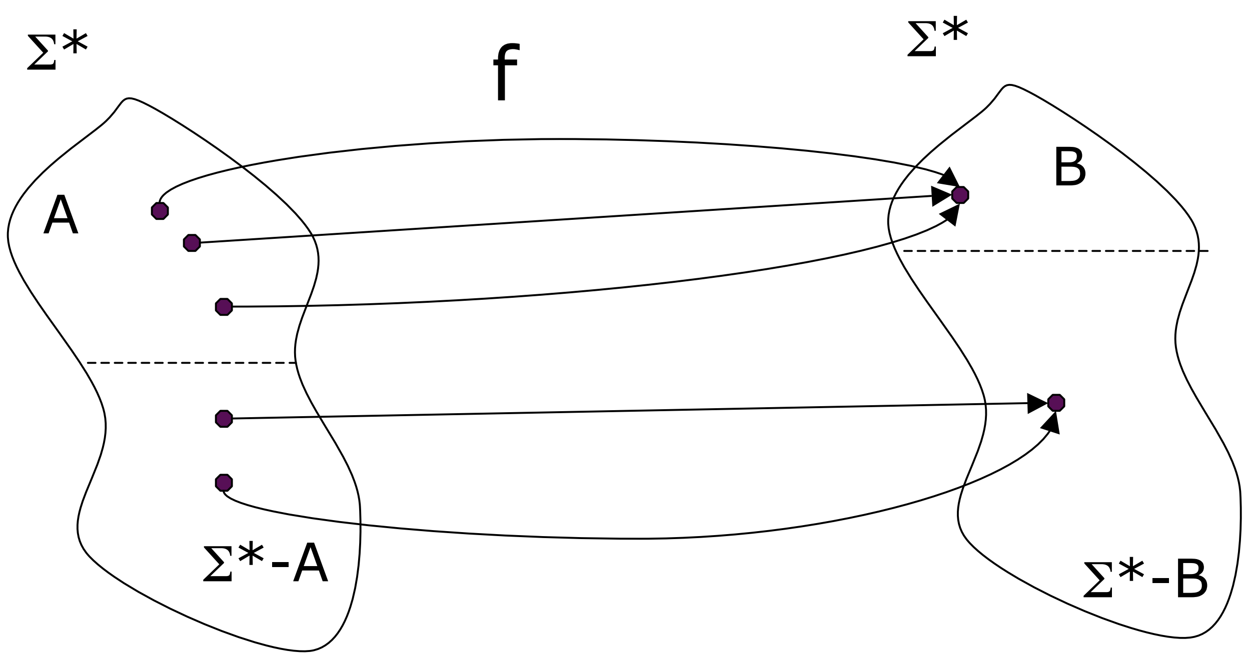
- $A \leq_m B$ implies if B were computable, so would A be:
  - To determine if x is in A: compute f(x) by machine, determine whether f(x) is in B, to get your answer.
- It also means the contrapositive, that if A is uncomputable, so is B.

Example

- $K \leq_m A^c_{TM}$
- $K = \{<M> \mid <M> \notin L(M)\}$
- $A^c_{TM} = \{<M, w> \mid w \notin L(M)\}$
- The reduction mapping in this case is:

  $$f(<M>) = <M, M>$$

$\leq_m$ Mapping need not be 1-1



How about $HALT_{TM} \leq A_{TM}$?

- Here we can use a mapping reduction.
- Assume A_{TM} is decidable. Map $HALT_{TM}$ into it.

- Given a machine and input <M, w>, map it to <M', w> by creating <M'> from <M>:
  - Change all **halting** (accepting or rejecting) states of M into **accepting** states of M', leaving everything else unchanged.
  - The mapping reduction f does this transformation: f(<M, w>) = <M', w>.
  - So M halts on w iff M' accepts w.
  - i.e. <M, w> $\in HALT_{TM}$ iff <M', w> $\in A_{TM}$.

Emptiness Problems

- E_{DFA} = {<B> | B is a DFA accepting no strings}
- E_{REX} = {<R> | R is a regular expression and L(R) = ∅}
- E_{CFG} = {<G> | G is a context-free grammar and L(G) = ∅}.
- E_{CSG} = {<G> | G is a context-sensitive grammar and L(G) = ∅}.
- E_{TM} = {<M> | M is a Turing machine and L(M) = ∅}
- Which of these are decidable? (Not necessarily obvious).

E_{TM} is not decidable

- Use a mapping reduction $A_{TM} \leq_m E^c_{TM}$, the latter accepting those <M> where L(M) is **non-empty**, in other words M accepts ***some*** input.

- Transformation: With input <M, w>, construct machine <M'> such that:
  - <M'> $\in E^c_{TM}$
    iff
  - <M, w> $\in A_{TM}$

- M': With input x, erase x, write w on the tape, then behave as M.

Property of M′

- M′ does exactly the same thing on every input x.
  - It accepts x iff M accepts w.
  - It rejects x iff M rejects w.
  - It diverges on x iff M diverges on w.

- So it accepts *some* input iff M accepts w.

Is either E_{TM} or its complement recognizable?

$A \leq_m B$ and recognizability

- $A \leq_m B$ also implies that if B were recognizable, so would A be.

- To recognize if x is in A: compute f(x) by machine, ask whether f(x) is in B.

  If the answer is affirmative, then the original x was in A.

  If we never get an answer, then we don't get answer for whether the original x was in A either.

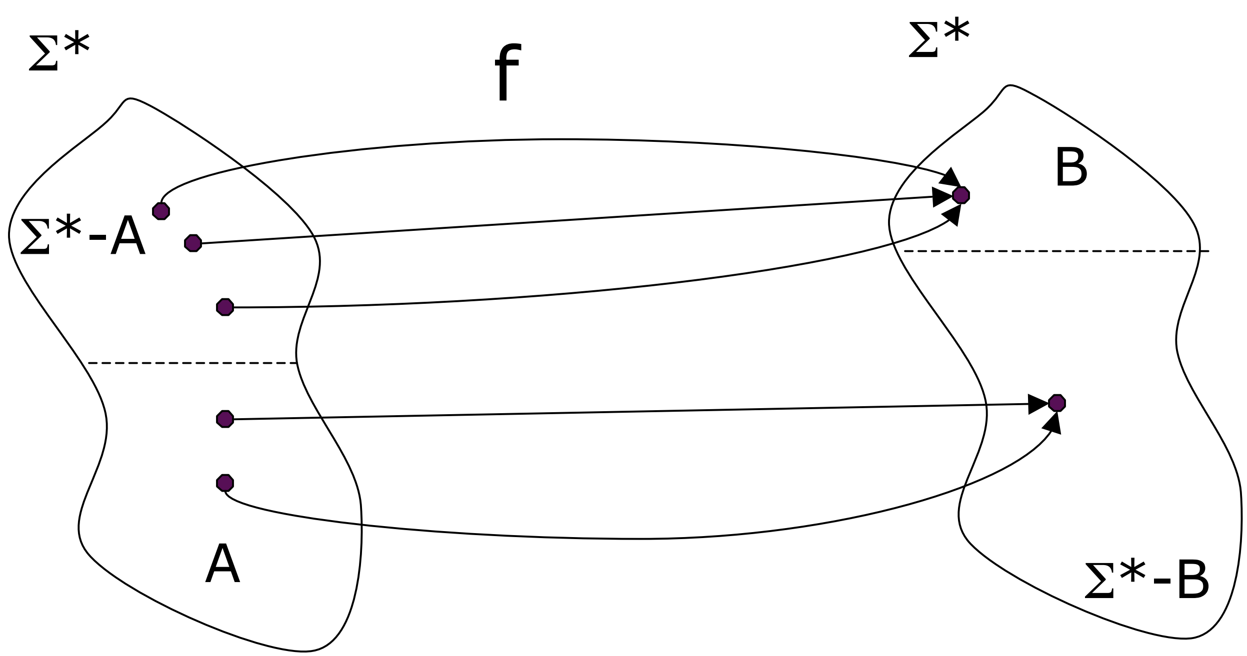
- implies if A is *not* recognizable, neither is B.

$A \leq_m B$ and co-recognizability

- $A \leq_m B$ similarly implies that if B were co-recognizable, so would A be.
- $A \leq_m B$ implies if A is *not* co-recognizable, neither is B.

$A^c \leq_m B$

- $A^c \leq_m B$ means the complement of A reduces to B. This is **equivalent** to $A \leq_m B^c$.



Sipser Problem 5.7

- If A is recognizable, and $A \leq_m A^c$, then A is decidable.

- Proof: If $A \leq_m A^c$, then also $A^c \leq_m A$, per the previous slide. But since A is recognizable, by $A^c \leq_m A$, A^c is also recognizable. But then both A and are A^c recognizable, so A is decidable from the complementarity lemma.

Example

- We already showed $A_{TM} \leq_m E^c_{TM}$.
- In effect, the same mapping gives $A^c_{TM} \leq_m E_{TM}$.
- But A^c_{TM} is not recognizable.
- Therefore E_{TM} is not recognizable.
- However, E^c_{TM} is co-recognizable.

“ALL” Problems

- ALL_{DFA} = {<B> | B is a DFA accepting all strings}
  (“all” means all of Σ^* where Σ is the alphabet of the DFA)
- ALL_{REX} = {<R> | R is a regular expression and
  $L(R) = \Sigma^*$}
- ALL_{CFG} = {<G> | G is a context-free
  grammar and $L(G) = \Sigma^*$}.
- ALL_{CSG} = {<G> | G is a context-sensitive
  grammar and $L(G) = \Sigma^*$}.
- ALL_{TM} = {<M> | M is a Turing machine and $L(M) = \Sigma^*$}
- Which of these are decidable? (Not necessarily obvious).

ALL_{TM}

- Is ALL_{TM} decidable?
- What is your intuition?
- How would you prove it?
- (Remember the tune "All or Nothing At All".)
- What about recognizable or co-recognizable, if not decidable.

Fixed-String Acceptance

- Suppose x_0 is a fixed string, such as the empty string (blank tape).
- Is there an algorithm for determining whether an arbitrary TM accepts x_0?
- We call the corresponding language acceptance problem **Accepts-$\mathbf{x_{0\ TM}}$**.

Accepts-$x_{0\ TM}$ is unsolvable, for any fixed x_0.

- See if you can reduce one of the other unsolvable problems to this problem.

$Regular_{TM}$ is Undecidable

- Define <M>$\in$ $Regular_{TM}$ iff L(M) is regular.

- Reduce a known unsolvable problem to $Regular_{TM}$.

- Try a mapping reduction.

Reduction to Regular_{TM}

- Reduce A_{TM} to Regular_{TM} as follows:
- Given an instance <M, w>, create <M'> as follows:
  - M' with input x: check whether x has the form 0^n1^n for some n.
  - If x does not have the form, reject.
  - If x does have the form, run M on w, accepting when and if M accepts w. (This may, of course, diverge, in which case M cannot accept w.)

Reduction to Regular_{TM}

- Thus M′ accepts exactly strings of the form 0^n1^n iff M accepts w. Otherwise M accepts $\varnothing$.

- So L(M′) is either
  - regular ($\varnothing$) or
  - non-regular ($\{0^n1^n \mid n \geq 0\}$)

  depending on whether M accepts w.

  [We've thus shown $A_{TM} \leq_m \text{Regular}^c_{TM}$.]

  What about Regular itself?

Functional vs. Structural Properties

- We have been using TM's to represent languages.
- When a property of a machine depends only on the **language** and not the specific TM used to represent the language, we call this a **functional property**.
- When a property depends on the **specific** details of a **machine**, it is called a **structural property**.

Functional vs. Structural Examples

- Functional:
  - L(M) = ∅
  - L(M) is regular
  - L(M) is infinite
  - L(M) is decidable
  - L(M) is recognizable (always true, by definition)

- Structural:
  - M has more than 100 reachable control states
  - M writes a non-blank character on its tape when started on an empty tape.
  - M reverses its direction of head travel at least once on each input.

Decidable vs. Undecidable Structural Properties

- M has more than 100 control states.
- M reaches a specific control state when started on an empty tape.
- M writes a non-blank character on its tape when started on an empty tape.
- M uses more than a specified amount of tape when started on a certain input.

- Which of these properties is decidable?

Rice’s Theorem, Informally

- Practically **no** functional properties are decidable for TM recognizable languages.

- The caveat here is that the theorem only applies to functional properties, not structure ones.

Trivial Functional Properties

- A functional property of a recognizable language is called **trivial** if it is either:
  - true for *no* recognizable language, or
  - true for *all* recognizable languages

- A **non-trivial property**, then, holds for **some** recognizable languages, but **not all**.

Rice’s Theorem

- **Any non-trivial functional property of the recognizable languages is not decidable.**

- Put another way:

  For any non-trivial functional property, there is no algorithm that will determine whether or not the language recognized by a given TM has the property.

Observations about Non-Trivial Properties P

- Any given language has property P, or it does not.
- A language L has property P iff L does not have $\neg$P.
- To decide P, it is adequate to decide the complementary property $\neg$P, since yes/no answers are required in both cases.

Proof of Rice’s Theorem (1 of 3)

- Suppose **P** is a non-trivial property.
- **Critical assumption**: **The empty language ∅ does not have property P**. If the opposite is true, then interchange P and ¬P so that the assumption is true. [This would be necessary in the case of P = Regular, for example.]
- Let $\mathbf{L_P}$ be some arbitrary recognizable language **with** property P (which must exist, because P is non-trivial). We know that $\mathbf{L_P}$ is distinct from ∅, by the assumption above. Let $\mathbf{M_P}$ be a machine recognizing the chosen language $\mathbf{L_P}$.
- **Plan**: Reduce the acceptance problem to that of deciding property P, which will imply that there is no algorithm for the latter. This will hinge on having the P decider **differentiate** between $\mathbf{L_P}$, which has property P, and ∅, which does not.

Proof of Rice’s Theorem (2 of 3)

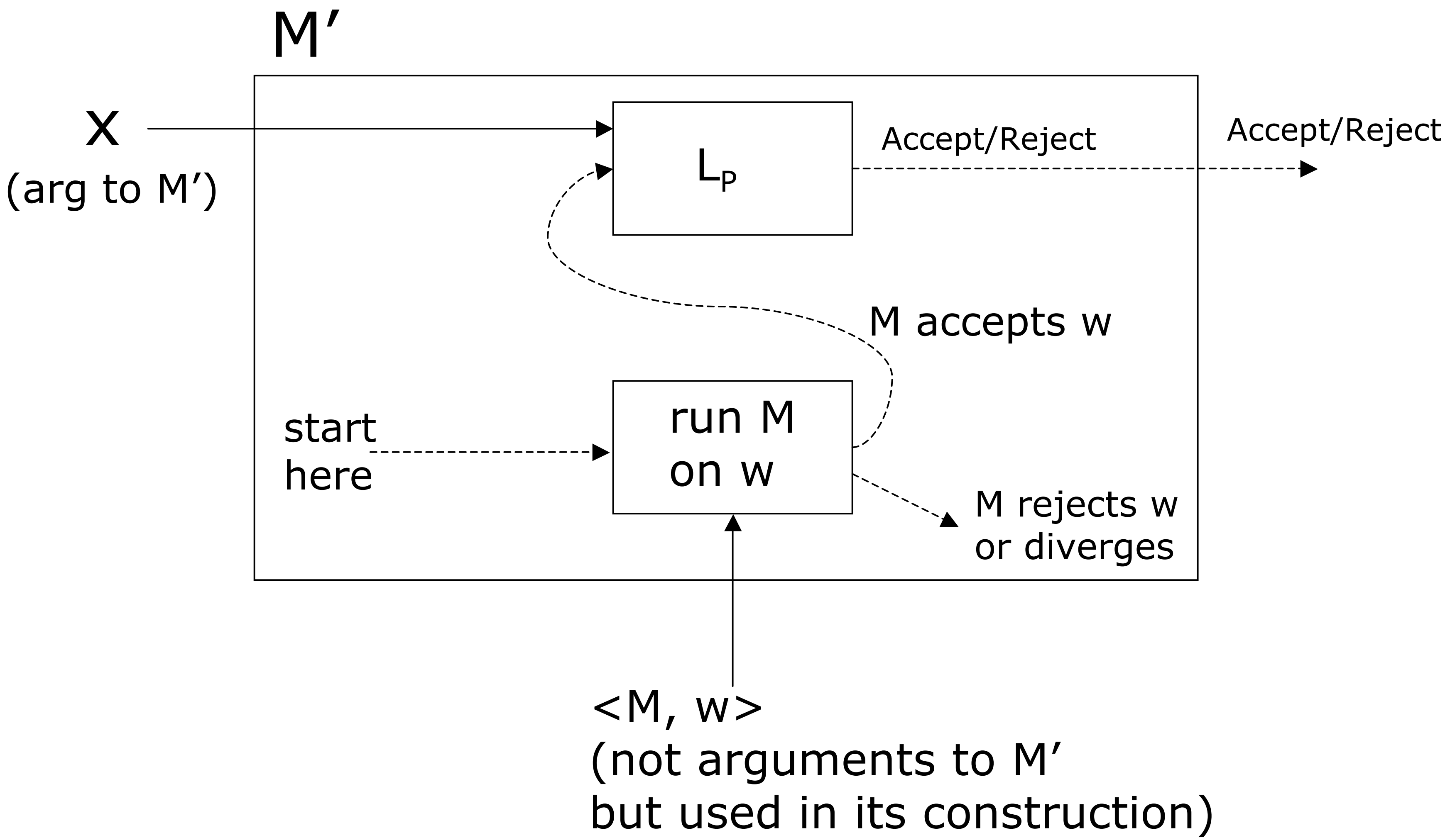
- The reduction works as follows: Suppose we have an algorithm for testing property P for any input <M>.

- We can then test whether an arbitrary Turing machine M accepts an arbitrary input w by constructing a machine M’ with the specifications on the following page.

Proof of Rice's Theorem (3 of 3)

- M': with input **x**, (temporarily set aside x and) start **behaving as** M on w.
- **If** M on w **accepts**, then continue by behaving as $\mathbf{M_P}$ on the original input x.
- Thus if and when M accepts w, M' will accept x iff $\mathbf{M_P}$ accepts x, i.e. L(M') = L($\mathbf{M_P}$). So in this case, L(M') **has** property P, because $\mathbf{M_P}$ was selected to have it.
- If M on w **does not terminate**, then L(M') = ∅, which by the critical assumption, **does not** have property P.
- So if we can test an arbitrary machine whether its language has property P, we can test M' in particular. But the answer to this test determines whether or not M accepts w.

Diagram of M′ as constructed
(dashed lines = control, solid = data)



Note

- Although P was discussed as a property of **language** recognized by a Turing machine, the same proof works in the case that:

  P is a property of the **partial function** computed by a Turing machine.

- This means, for example, there is no algorithm that will decide equivalence of an arbitrary machine's partial function to that of a given machine.

Languages neither recognizable nor co-recognizable.

- EQ_{TM} is neither.

- Proof that **EQ_{TM} is not recognizable**:
  Use reduction from A_{TM} (which is not co-recognizable) to EQ^c_{TM}. Given <M, w>, construct M_1 and M_2 thus:
  - M_1 **always rejects**.
  - M_2 on any input, behaves as M on w.
- So $<M_1, M_2> \notin EQ_{TM}$ iff $<M, w> \in A_{TM}$.

Languages neither recognizable nor co-recognizable.

- EQ_{TM} is neither.

- Proof that **EQ^c_{TM} is not recognizable**:
  Use reduction from A_{TM} (which is not co-recognizable) to EQ_{TM}. Given <M, w>, construct M_1 and M_2 thus:
  - M_1 **always accepts**.
  - M_2 on any input, behaves as M on w.
- So $<M_1, M_2> \in EQ_{TM}$ iff $<M, w> \in A_{TM}$.

Languages neither recognizable nor co-recognizable.

- ALL_{TM} is neither.
- To show ALL_{TM} is **not co-recognizable**, give a mapping reduction from A_{TM} (which is not co–recognizable), to ALL_{TM}.

  $f(\langle M, w\rangle) = M'$, where $M'(x) = M(w)$.

  Thus M’ accepts all iff M accepts w.

- But how to show ALL_{TM} is not recognizable??

ALL_{TM} is not recognizable

- We prove this by $HALT^c_{TM} \leq_m ALL_{TM}$.
- Assume that ALL_{TM} *is* recognizable.
- Let <M, w> be an arbitrary machine with input. The reduction mapping constructs M' to behave as follows:
  - M'(x) simulates M(w) for |x| steps.
  - If M(w) **fails to halt** within |x| steps, accept x.
  - Otherwise reject x.
- Hence M' accepts all inputs iff M(w) does not halt.