Your name ___________________________

Harvey Mudd College
**CS 81 Mid-Term Exam**
Spring semester, 2010

- Please write legibly.
- You may use a single 2-sided 8.5” x 11” reference sheet, but no other reference material.
- Please turn in your reference sheet along with the exam. It will be returned to you.
- The exam has a 75-minute overall time limit. Do the best you can in that time. It is best to spend your time working on problems that pay off in terms of the most points.
- Note that some questions say “prove”, while others say “prove or disprove”, leaving it up to you to decide which is possible.

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1. [10 points]

   Prove the two sequents below using the Sequent Calculus. (If unable to do it with the Sequent Calculus, you may use an alternate method for half-credit.) The formulas are repeated at the bottom of the space for your convenience.

   Part I: $r \rightarrow \neg(p \vee q) \vdash (p \rightarrow \neg r) \wedge (r \rightarrow \neg q)$

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Part II (using Sequent Calculus): $(\exists x\ R(x)) \rightarrow S(a) \vdash (\forall y\ \neg R(y)) \vee (\exists z\ S(z))$

$(\exists x\ R(x)) \rightarrow S(a) \vdash (\forall y\ \neg R(y)) \vee (\exists z\ S(z))$

2. [10 points] Prove or disprove each sequent, using any method you wish.
   Part I:

   $\exists x\, P(x),\ \exists x\, (P(x) \rightarrow Q(x)) \vdash \exists x\, Q(x)$

   Part II:

   $\forall x\, \exists y\, P(x, y),\ \forall y\, \exists x\, P(x, y) \vdash \exists x\, P(x, x)$

3. [20 points] Prove formula d. below, using natural deduction and the Peano axioms (a, b, c). Recall that $\varphi[t/x]$, where t is a term, means $\varphi$ with t replacing each free occurrence of x.

   a. $\forall x \neg(s(x) = 0)$
   b. $\forall x \forall y\ (s(x) = s(y) \rightarrow x = y)$
   c. [Induction axioms] For any formula $\varphi$: $(\varphi[0/x] \wedge \forall x\ (\varphi \rightarrow \varphi[s(x)/x])) \rightarrow \forall x\ \varphi$

   d. To be proved: $\forall x \neg(s(x) = x)$.

4. [15 points]

Carefully prove using natural deduction:

$\exists x\, P(x) \land \exists x\, Q(x) \vdash \exists x\, \exists y\, (P(x) \land Q(y))$

5. [15 points]

Translate each of Parts I and II below into one or more formulas, then prove or disprove each part, using any method (the premises of both parts are the same, so you only have to provide one translation of them). Provide any insightful comments that connect the two parts.

Part I:

a. John loves a person iff that person does not love him or herself.

b. John is a person and Mary is a person.

c. Mary loves herself.

Therefore d. John does not love Mary.

Part II:

a. John loves a person iff that person does not love him or herself.

b. John is a person and Mary is a person.

c. Mary loves herself.

Therefore d. John loves Mary.

6. [20 points]

Using Hoare Logic, and any relevant algebraic identities (which need not be proved), prove the partial correctness of the following triple:

```
{n ≥ 0}

x := 0; ___________________
y := 1; ___________________
while x < n do
      y := y + y; _________
      x := x + 1  _________
      od

{y = 2^n}
```

Suggestion: Show the assertions between each line off to the right (where the lines are), then establish the relevant triples.

7. [10 points]

Using the clauses below and resolution, prove that 2+2 = 4 and 2×2 = 4, where number 2 is represented as `s(s(0))`, and the number 4 is represented as `s(s(s(s(0))))`, where `s` is the successor function.

a(x, y, z) means that z is the sum of x and y, and m(x, y, z) means that z is the product of x and y. The following clauses derive directly from the number theory axioms.

1. a(0, x, x).
2. a(x, y, z) ⟶ a(s(x), y, s(z)).
3. m(0, x, 0).
4. (m(x, y, z) ∧ a(y, z, w)) ⟶ m(s(x), y, w).